Fractal Functions of Discontinuous Approximation

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Abstract: A procedure for the definition of discontinuous real functions is developed, based on a fractal methodology. For this purpose, a binary operation in the space of bounded functions on an interval is established. Two functions give rise to a new one, called in the paper fractal convolution of the originals, whose graph is discontinuous and has a fractal structure in general. The new function approximates one of the chosen pair and, under certain conditions, is continuous. The convolution is used for the definition of discontinuous bases of the space of square integrable functions, whose elements are as close to a classical orthonormal system as desired.

Keywords: Discontinuous Functions, Interpolation, Approximation, Functional Spaces, Fractals.

INTRODUCTION

Since the advent of the fractal theory (and its predecessors) the emphasis has been on continuous functions (from Weierstrass map to wavelets), being the discontinuous case minimal. In fact there are only a few examples of mathematical models with discontinuities (Haar basis [1], for instance), deserving a deeper and additional study. In this paper we develop a procedure for the definition of discontinuous real mappings, based on a fractal methodology.

We set a binary operation in the space of bounded functions on a compact interval $\mathcal{B}(I)$ (although the construction can be done in the space of measurable essentially bounded mappings $\mathcal{L}^\infty(I)$ as well). Two functions (f and b) give rise to a new one $f*b$ (called in this paper fractal convolution of the originals), whose graph has a fractal structure in general.

The convolution is defined as fixed point of a nonlinear contractive operator T of $\mathcal{B}(I)$, where the images Tg are piecewisely defined by means of a partition of the interval and a scale function. The convolution may interpolate one of the initial pair, if additional conditions on the nodes of the partition are imposed. With more stringent requirements, the result $f*b$ is continuous.

In the last part of the text, the convolution is used for the definition of discontinuous bases of the space of square integrable functions $\mathcal{L}^2(I)$, close to a classical orthonormal system if desired.

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2. A FRACTAL CONVOLUTION

Let us consider the space $\mathcal{B}(I)$ of bounded functions on a compact interval $I=[c,d]$ and functions $f,b,\alpha \in \mathcal{B}(I)$ such that $\|\alpha\|_\infty < 1$, where

$$\|\alpha\|_\infty = \sup\{|\alpha(t)|:\ t \in I\}.$$

The mapping α will be called scale function. Let Δ : $a=t_0<t_1<...<t_N=b$ be a partition of the interval I, and $L_n = a_n t + b_n$ contractive affinities such that $L_n(t_0)=t_{n-1},\ L_n(t_N)=t_n$.

Let us consider fixed f,b,α and Δ with the conditions prescribed and define the operator $T=T_{f,b,\alpha,\Delta}$: $\mathcal{B}(I)\to\mathcal{B}(I)$, defined as

$$Tg(t) = f(t) + \alpha(t)(g-b)\circ L_n^{-1}(t), \qquad (1)$$

for $t \in I_n$, $n=1,2,...,N$. The intervals I_n are defined as $I_n=(t_{n-1},t_n]$ for $n=2,...,N$, and $I_1=[t_0,t_1]$. Let us see that T is a contraction defined on the Banach space $(\mathcal{B}(I),\|\cdot\|_\infty)$. This is proved considering that, if $g,g' \in \mathcal{B}(I)$, and $t \in I_n$,

$$|Tg(t)-Tg'(t)| = |\alpha(t)||(g-g')\circ L_n^{-1}(t)|,$$

and thus

$$\|Tg-Tg'\|_\infty \le \|\alpha\|_\infty \|g-g'\|_\infty .$$

Consequently, T admits a unique and contracting fixed point $\tilde{f}$.

Definition 2.1. The map $\tilde{f}=f*b$ is the fractal convolution of f and b with respect to the partition Δ and the scale function α.

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The convolution satisfies the fixed point equation:

$$f*b(t)=f(t)+\alpha(t)(f*b-b)\circ L_n^{-1}(t), \qquad (2)$$

for $t\in I_n$. This is a self-similar expression that endows the graph of $f*b$ with a fractal structure.

If one considers a piecewise constant $\alpha(t)=\alpha_n$ for the n-th subinterval, f,b continuous on I and such that $f(t_0)=b(t_0)$, $f(t_N)=b(t_N)$ then $f*b$ interpolates f on the nodes of the partition, and $f*b=f^\alpha$ is the α-fractal function of f with respect to b and Δ, defined in previous papers (see for instance [2, 3]).

In the general case, $f*b$ is bounded, discontinuous and not interpolatory (see Figure **1**). If one wishes a certain approximation ε of f at the nodes, one must choose b to be close to f at the extremes of the interval according to the following result.

Proposition 2.2. The distance between f and its fractal convolution does not exceed ε on the nodes of the partition, that is to say,

$$|f*b(t_n)-f(t_n)|\le\varepsilon,$$

for all $n=0,1,2,\ldots,N$ if $\max\{|f(t_i)-b(t_i)|:\ i=0,N\}\le\varepsilon(1-\|\alpha\|_\infty)$.

Proof. Using the fixed point equation (2) and the definition of L_N:

$$f*b(t_N)=f*b(L_N(t_N))=f(t_N)+\alpha(t_N)(f*b(t_N)-b(t_N)),$$

then

$$\left|f*b(t_N)-f(t_N)\right|\le\|\alpha\|_\infty\left(\left|f*b(t_N)-f(t_N)\right|\right)+\left|f(t_N)-b(t_N)\right|),$$

and

$$\left|f*b(t_N)-f(t_N)\right|\le\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\left|f(t_N)-b(t_N)\right|. \qquad (3)$$

For a node t_n such that $1\le n\le N$, the fixed point equation (2) and the definition of L_n provides:

$$f*b(t_n)=f*b(L_n(t_N))=f(t_n)+\alpha(t_n)(f*b(t_N)-b(t_N)),$$

then

$$\left|f*b(t_n)-f(t_n)\right|\le\left|f*b(t_N)-b(t_N)\right|.$$

According to (3),

$$\left|f*b(t_n)-f(t_n)\right|\le\left|f*b(t_N)-f(t_N)\right|+$$
$$\left|f(t_N)-b(t_N)\right|\le\frac{1}{1-\|\alpha\|_\infty}\left|f(t_N)-b(t_N)\right|, \qquad (4)$$

and bearing in mind the condition on ε,

$$\left|f*b(t_n)-f(t_n)\right|\le\varepsilon.$$

Similar arguments provide the inequality

$$\left|f*b(t_0)-f(t_0)\right|\le\frac{1}{1-\|\alpha\|_\infty}\left|f(t_0)-b(t_0)\right|\le\varepsilon. \qquad (5)$$

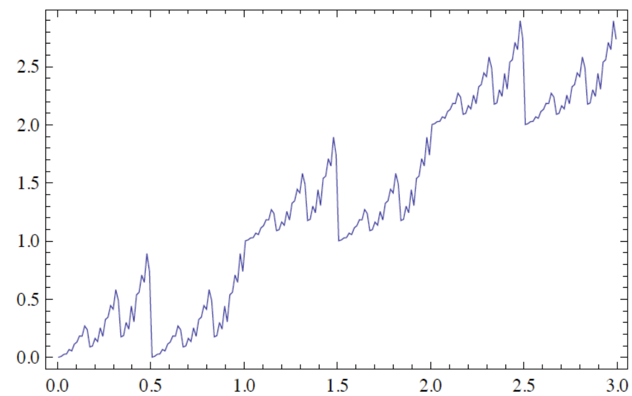

Figure 1: Graph of the fractal convolution of integer part of $t(f(t))$ and $b(t)=t+0.3$, with respect to a uniform partition with N = 6, and scale function $\alpha(t)=-t/4$ in the interval $I=[0,3]$. The vertical lines display discontinuities.

Consequence: If $f(t_0)=b(t_0)$ and $f(t_N)=b(t_N)$, then $f*b$ interpolates f at the nodes of the partition ((4), (5)), but the convolution need not be continuous.

In general, the distance between $f*b$ and f is modulated by the distance between f and b, and the amplitude of the scale function α, according to the following inequality:

$$\|f*b-f\|_\infty\le\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-b\|_\infty, \qquad (6)$$

that is a consequence of the equation (2). As a particular case, if either α is the null function or $f=b$ then $f*b=f$.

Let us consider now the binary operation $\mathcal{P}=\mathcal{P}_{\Delta,\alpha}:\mathcal{B}(I)\times\mathcal{B}(I)\to\mathcal{B}(I)$ defined by

$$\mathcal{P}(f,b)=f*b,$$

with respect to a given partition Δ and scale function $\alpha \in \mathcal{B}(I)$. The operation is idempotent since $f * f = f$ for any $f \in \mathcal{B}(I)$. The product space $\mathcal{B}(I) \times \mathcal{B}(I)$ is endowed with the norm:

$$\|(f,b)\|_\infty = \|f\|_\infty + \|b\|_\infty . \tag{7}$$

Proposition 2.3. $\mathcal{P}$ is a linear and bounded operator.

Given $f, f`, b, b` \in \mathcal{B}(I)$, considering the equations type (2) relative to $f * b$ and $f`*b`$, and adding them, we observe that the function $(f*b)+(f`*b`)$ satisfies the fixed point equation that corresponds to $(f+f`)*(b+b`)$. Its unicity implies that $(f+f`)*(b+b`) = (f*b)+(f`*b`)$ and thus $\mathcal{P}(f+f`, b+b`) = \mathcal{P}(f,b) + \mathcal{P}(f`,b`)$. An argument similar holds for $\lambda(f,b)$, and thus $\mathcal{P}$ is linear.

The expression (6) implies that

$$\|f*b\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-b\|_\infty + \|f\|_\infty ,$$

$$\|f*b\|_\infty \le \frac{1}{1-\|\alpha\|_\infty}\|f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|b\|_\infty \le \frac{\|f\|_\infty + \|b\|_\infty}{1-\|\alpha\|_\infty}.$$

Thus, the operator is continuous and satisfies the inequality

$$\|\mathcal{P}\| \le \frac{1}{1-\|\alpha\|_\infty},$$

where $\|\mathcal{P}\|$ is the operator norm with respect to the norm (7) in the product space.

The former arguments hold in the framework of the measurable essentially bounded function space $\mathcal{L}^\infty(I)$ as well. From here on we assume that $f, b, \alpha \in \mathcal{L}^\infty(I)$, and $\|\alpha\|_\infty < 1$, and define the mean of order p norm (or p -norm)

$$\|f\|_p = \left(\int_I |f|^p \, dt\right)^{1/p}.$$

The expression (6) can be extended to the p - norm.

Proposition 2.4. If $1 \le p \le \infty$, the following inequality holds.

$$\|f*b-f\|_p \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-b\|_p \tag{8}$$

Proof. It is similar to the proof of Proposition 3 of [3].

3. DISCONTINUOUS BASES OF $\mathcal{L}^2(I)$

The previous constructions allow us the definition of fractal bases of the space of square-integrable functions on I. The elements of the new spanning systems will be discontinuous in general, and as close to a classical basis as desired.

Let $(p_m)_{m=0}^\infty$ be an orthonormal basis of $\mathcal{L}^2(I)$ such that $p_m \in \mathcal{L}^\infty(I)$ for all $m = 0,1,...$ For instance p_m may be normalized Legendre polynomials on $I = [-1,1]$.

Let us consider $b \in \mathcal{L}^\infty(I)$ and a partition of the interval. If $f \in \mathcal{L}^2(I)$, f admits a unique expression in terms of the basis elements

$$f = \sum_{m=0}^\infty c_m p_m . \tag{9}$$

The orthonormality implies that

$$c_m = \langle f, p_m \rangle,$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product of $\mathcal{L}^2(I)$. The inequality of Cauchy-Schwarz provides a bound for the coefficients

$$|c_m| = |\langle f, p_m \rangle| \le \|f\|_2 \|p_m\|_2 = \|f\|_2 .$$

Let us choose $K = 1 + \|b\|_2$ and a sequence of scale functions $\alpha_m(t)$ for $m = 0,1,...$ (in order to define $p_m * b$) such that the sum

$$A = \sum_{m=0}^\infty \frac{\|\alpha_m\|_\infty}{1-\|\alpha_m\|_\infty}$$

satisfies the expression $AK \le r$ for some $r < 1$. Let us define the operator $S : \mathcal{L}^2(I) \to \mathcal{L}^2(I)$

$$S(f) = \sum_{m=0}^\infty c_m (p_m * b),$$

where c_m are the coefficients of f (9). Then

$$(I-S)(f)=\sum_{m=0}^{\infty} c_m (p_m - p_m * b),$$

verifies (using (8), (10) and the properties of the norm)

$$\|(I-S)(f)\|_2 \le \sum_{m=0}^{\infty} |c_m| \|p_m - p_m * b\|_2 \le \|f\|_2 KA \le r\|f\|_2 .$$

Then $\|I-S\|_2 < 1$ and S is invertible. S is an isomorphism and $(p_m * b)_{m=0}^{\infty}$ is a basis of $\mathcal{L}^2(I)$ (further details can be read in [2], Th. 4.9).

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