An Analytical Solution to Heat Equation

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Abstract: The present paper explores theoretical aspects of solving the heat equation in the monodimensional case. The proposed study method is the variables separation method, finding thus an original solution, which is compared with the numerical solution obtained by finite difference method.

Keywords: Heat equation, non-stationary regime, variables separation method, finite difference method.

1. INTRODUCTION

The heat equation is a parabolic partial differential equation of importance in various scientific fields [1-3]. Thus, it is linked to the study of Brownian motion *via* the Fokker-Planck equation or diffusion equation. The latter is a more general version of the heat equation used in the study of chemical diffusion or other related processes, for example, corrosion. The equation describes temperature variation over time, in a given region. Analytical methods allow determining the temperature in any point of interest in the study environment. By contrast with analytical methods, numerical methods permit determining the numerical value of temperature only in discrete points [4]. More and more different geometric based approaches are reported. In their case, point selection is done by dividing the region on interest in a multitude of smaller areas, and assigning these a reference point (called node), represented by the center of this area. Nodes are represented by a matrix, called network of nodes. Coordinates, denoted by x and y, are determined by indices of this matrix, denoted by m and n. Calculations can be performed faster than the physical process considered actually occurs, which is a major advantage in optimizing technological processes. There are few restrictions on time steps to ensure stability of the numerical method [5]. The method allows expression of temperature at time $t+\Delta t$, as a function of their temperatures at time t.

2. STATING THE HYPOTHESES

The heat equation describes the process of conduction of heat through a function $w(x,y,z,t)$

$$\frac{\partial w}{\partial t} - D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0\ , \tag{1}$$

with one-dimensional form, $w = w(x,t)$, in a bar of length l

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}\ , \tag{2}$$

where, D is a constant coefficient of heat conduction, called thermal diffusibility, and the solution is obtained only for a finite time, t_f , requiring at limit specifications at $x=0$ and $x=l$, but also at initial conditions $t=0$, such as $w(0,t) = w\ 0$, $w(l,t) = wl$ and $w(x,0) = \varphi\ (x)$.

The equation can also be found in the form

$$w_{t(x,t)} = d^2 w_{xx}(x,t)\ ,\ 0 \le x < l \text{ and } 0 < t < t_f\ ,$$

and models temperature distribution in an insulated bar, having its ends maintained at constant temperatures, d_1, and d_2, with the initial temperature distribution given by the function $\varphi(x)$, under initial conditions $w(x,0) = \varphi(x)$, for $t=0$ and $0 \le x \le l$, and at limit conditions $w(0,t) = f_1(t) \equiv c_1$, for $x=0$ and $0 \le t \le t_f$ and $w(l,t) = f_2(t) \equiv c_2$, for $x=l$ and $0 \le t \le t_f$.

3. ANALYTICAL SOLUTION

The study of heat propagation through a finite bar, under non-steady state assumptions, leads to mathematical formulation of partial differential equations of type (2) with at limit conditions

$$w(0,t) = w(l,t) = 0\ , \tag{4}$$

and initial conditions

$$w(x,0) = \varphi(x)\ . \tag{5}$$

At limit, the ends of the bar have a temperature equal to zero, otherwise the bar temperature is expressed with the not-null function $\varphi(x)$, continuous

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on the interval [0, l], situation in which the following condition also needs to be fulfilled

$$\varphi(0)=0 . \quad (6)$$

To find the solution of partial differential equation using the separation of variables method in conjunction with the superposition principle [6], we will seek a solution of the form

$$w(x,t)=X(x)T(t) . \quad (7)$$

By deriving it, then separating the derived variables, and noting common value with μ , we obtain

$$\frac{X''(x)}{X(x)}=\frac{1}{D}\frac{T'(t)}{T(t)}=\mu, \quad (8)$$

which only has solutions for $\mu<0$.

Thus, when the temperature increases, it could exceed any positive value. If μ is equal to, T variable would in fact be a constant, therefore, with a change of variable:

$$\mu=-\lambda^2 , \quad (9)$$

$$T(t)=e^{-\lambda^2 Dt} . \quad (10)$$

The characteristic equation of the differential equation in X is

$$r^2+\lambda^2=0 , \quad (11)$$

with the general solution

$$X(x)=C_1 \cos \lambda x+C_2 \sin \lambda x . \quad (12)$$

From at limit conditions, we find $\lambda x=n\pi$, therefore an infinity of solutions

$$X_n(x)=C_{2n} \sin\frac{n\pi}{l}x, \quad (13)$$

and corresponding

$$T_n(t)=C_{1n}e^{-\frac{n^2\pi^2 D}{l^2}t} . \quad (14)$$

If the constants are accumulated $C_{2n}C_{1n}=A_n$, the resulting equation is

$$w_n(x,t)=X_n(x)T_n(t)=A_n e^{-\frac{n^2\pi^2 D}{l^2}t} \sin\tfrac{n\pi}{l}x . \quad (15)$$

Applying the superposition principle

$$w(x,t)=\sum_{n=1}^{8}A_n e^{-\frac{n^2\pi^2 D}{l^2}t} \sin\tfrac{n\pi}{l}x . \quad (16)$$

Determining A_n from the initial condition $w(x,0)=\varphi(x)$, with a given $\varphi(x)$

$$\varphi(x)=\frac{x(1-x)}{l^2}=\sum_{n=1}^{\infty}A_n \sin\frac{n\pi}{l}x \quad (17)$$

and extending the function $\varphi(x)$, $\varphi(x):[0,l]\to\mathbb{R}$, on the interval $[-l,0]$, through series development of sin functions:

$$A_n=\tfrac{2}{l}\int_0^l \varphi(x)\sin\tfrac{n\pi}{l}x . \quad (18)$$

The final solution of the heat equation for a finite homogenous bar is

$$\mathbf{w(x,t)=\sum_{n=1}^{8}\tfrac{2}{l}e^{-\frac{Dn^2\pi^2}{l^2}t}\sin\left(\tfrac{n\pi x}{l}\right)\int_0^l\varphi(x)\sin\left(\tfrac{n\pi x}{l}\right)dx} . \quad (19)$$

Solving the partial derivatives heat equation directly, an *original* solution is found, with separate variables of the form:

$$w(x,t)=\varphi 1(x)\,\varphi 2(t) \quad (20)$$

$$\varphi_1(x)=\left(C_1e^{\sqrt{c_1}x}+C_2e^{-\sqrt{c_1}x}\right), \quad (21)$$

$$\varphi_2(t)=C_1e^{Kc_1t} . \quad (22)$$

4. COMPARISON WITH NUMERICAL SOLUTION

Through finite difference method, the corresponding temperature of each node is the average temperature of the surrounding area. The selection of the points is done by taking into consideration geometrical restrictions, which depend on the number of these points. Assuming that the domain $Z=\{(x,t);\ 0\le x\le l,\ 0\le t\le t_f\}$ is divided in n-1 and m-1 rectangles with the sides measuring $\Delta x=h$ and $\Delta t=k$, then for $t=t_1=0$ the solution is $w(x_i t_1)=\varphi(x_i)$. The method for calculating the approximate solution $w(x,t)$ in successive rows of the network of nodes $\left\{w(x_i t_j);\ i=1,2,\ldots n\right\}$, for j=2,3,...m uses formulas with finite differences for calculating $w_{t(x,t)}$ and $w_{xx}(x,t)$

$$w_{t(x,t)} = \frac{w(x,t+k) - w(x,t)}{k} + R(k) \quad (23)$$

and

$$w_{xx}(x,t) = \frac{w(x-h,t) - 2w(x,t) + w(x+h,t)}{h^2} + R(h^2). \quad (24)$$

Grid spacing is uniform both on rows, $x_{i+1} = x_1 + h$ and $x_{i+1} = x_1 - h$, and on columns $t_{j+1} = t_j \quad +k$. Neglecting the terms $R(k)$ and $R(h^2)$ and using the approximation $w_{i,j}$ for $w(x_i, t_j)$ in the equations (3), which, being substituted, lead to a relationship approximating the solution of the equation:

$$\frac{w_{ij+1} - w_{ij}}{k} = d^2 \, \frac{w_{i-1j} - 2w_{ij} + w_{i+1j}}{h^2}. \quad (25)$$

For simplification, we will substitute $\mu = d^2 k / h^2$, thus obtaining the explicit relationship with finite difference

$$w_{i,j} = (1-2\mu) w_{i,j} + \mu \left(w_{i-1,j} + w_{i+1,j} \right). \quad (26)$$

This equation is also known as the finite difference method for the propagation of heat in one direction in non-stationary regime [4]. It is used to calculate the values from row (j+1), assuming that the values from row j are known. The relationship calculates explicitly the value of $w_{i,j+1}$ as a function of $w_{i-1,j}$ and $w_{i+1,j}$, but, the numerical method is stable only if the errors occurred in a calculation step diminish along the calculation process. Thus, the heat equation is stable if and only if $0 \le \mu \le 0.5$, which means that at step k the relationship $k \le h^2 / (2d^2)$ must be satisfied. If this condition is not fulfilled, the errors that occur in row $\{w_{i,j}\}$ can be amplified in a line $w_{i,p}$ for $p > j$. The accuracy of the solution of the differential equation depends on the rank $R(k)+R\ (h^2)$. The term $R(k)$ decreases linearly as k is close to 0. Thus, the results obtained in the network nodes, are not sufficiently accurate and both $\Delta x = h_0$, and $\Delta t = k_0$ must be decreased. Considering a new step $\Delta x = h_1 = h_{0/2}$, to obtain the same ratio μ, the new k must be

$$k_1 = \frac{\mu (h_1)^2}{d^2} = \frac{\mu (h_0)^2}{4d^2} = \frac{k_0}{4}. \quad (27)$$

As a consequence, a two-fold or four-fold increase of the number of points on the axis x, and t, but, usually, an eight-fold increase is prohibited and therefore more effective calculation formulas, without stability restrictions, must be used [7]. The analytical solution of the propagation of heat (Figure **1a**), in the monodimensional case, is compared with the solution obtained numerically (Figure **1b**). The simplicity of the numerical formulas makes them attractive for use, but attention must be given to the use of numerically stable methods.

5. CONCLUSIONS

The equation of heat propagation is a, relatively simple, parabolic equation, seen in the study of other

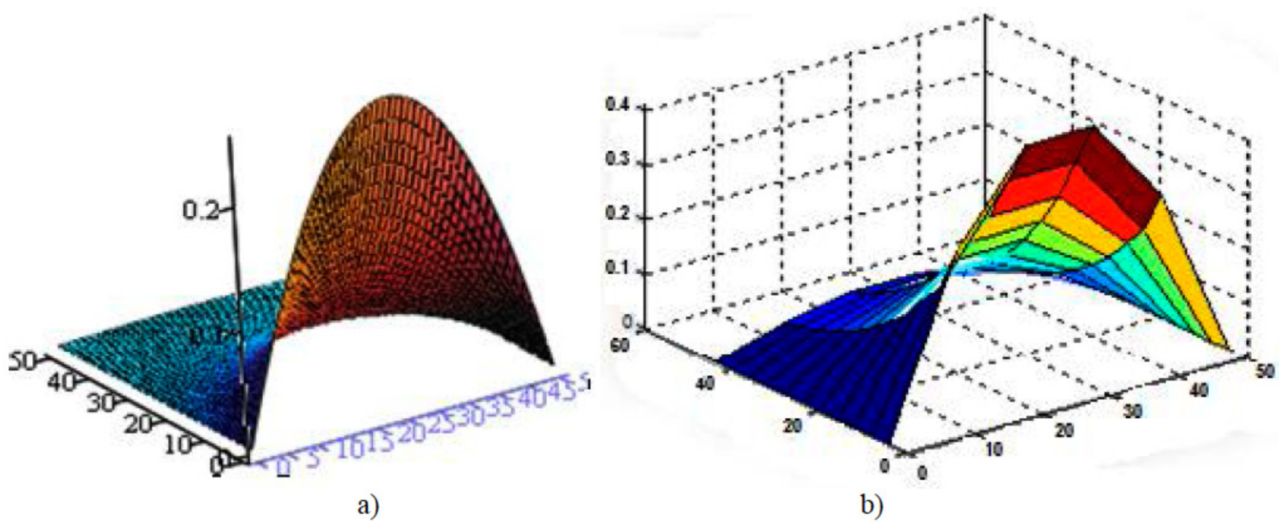

Figure 1: The solution of the heat equation, for $l = 50$ [mm], $t_f = 5\,10^{11}$ [s], $D = 10^{-8}$ (to compensate *D* between 10^{-5} *and* 10^{-15}), obtained through the method of: **a**) separating variables (analytically); **b**) finite difference, for φ (0,0,0,1,0.5,0.5,6,11) and μ =0.5 (numerically).

phenomena as well. The function w describes the temperature in a given location, and is a time-dependent function, as heat is propagated through space. The heat equation is used to determine changes over time in the function w. The numerical calculation method used to compare with the original solution discovered analytically allows the analysis of a domain with a geometric complexity often seen in practice, and the calculation accuracy is increased by comparison to other classical methods. If the number of nodes is high (a fine network) very accurate solutions can be produced. An interesting property is that, even though w is discontinuous at an initial time $t = t_0$, the temperature curve becomes smooth, the soonest for $t > t_0$.

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