A Piece of Paper and a Pair of Scissors

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Abstract: In this paper we will discuss the problem of splitting a given geometrical figure in two regions of equal area by drawing a line through a given point in the interior of the figure. For some geometric figures we will further discuss the possibility of splitting the figure in regions with areas p% and $(100-p)$ %.

Keywords: Areas, calculus.

INTRODUCTION

The problem is relatively simple in the case of some symmetrical geometric figures. For instance, in the case of a circle, given any point there is always a diameter through that point which will split the circle in two equal area regions. If the areas are not to be equal one could find, using methods of Calculus or Geometry, the line which will split the region in p% and $(100-p)$ % for certain values of p, which will depend on the distance from the center to the chosen point.

THE CIRCLE

As stated, the solution to splitting a circle in regions of 50-50% is to draw a diameter through the given point (a,b). All the results in the case of a circle are exactly as expected, however, we would like to prove these results using some Calculus methods. We can find the possible maximum and minumum areas obtained from a straight line drawn through this point by analyzing how the slope m of line l through (a,b) affects the resulting sectional areas.

In Figure **1**, a circle has been scaled so that it has a radius of 1, its center is $(0,0)$, and $(a,0)$ lies on the x-axis. The shortest distance between the center of the circle and the line $y=mx-ma$ through point $(a,0)$ is given by

$$d((0,0),l)=\frac{|ma|}{\sqrt{m^2+1}}.$$

The area of the sector determined by the line $y=m(x-a)$ is θ where 2θ is the smaller angle between the two radii, that means:

$$A_{sector}=\cos^{-1}\left(\frac{|ma|}{\sqrt{m^2+1}}\right)$$

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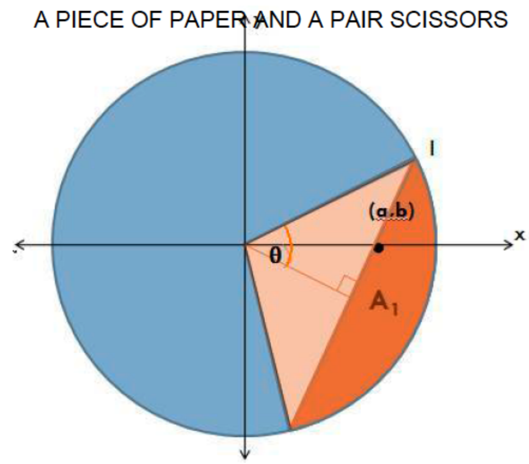

Figure 1: A circle, oriented so that a point $(a,0)$, taken arbitrarily, lies on the x-axis.

The area of the triangle nested within this sector is

$$A_{triangle}=\frac{|ma|}{\sqrt{m^2+1}}\sqrt{1-\left(\frac{|ma|}{m^2+1}\right)^2}$$

By subtracting the area of the triangle from the area of the sector, we obtain the area of one region created by the line. This area is equal to

$$A_1=\cos^{-1}\left(\frac{|ma|}{\sqrt{m^2+1}}\right)-\frac{|ma|\sqrt{m^2+1-m^2a^2}}{m^2+1}$$

Mathematica can be used to plot the function

$$A(a,m)=\frac{1}{\pi}\left[\cos^{-1}\left(\frac{|ma|}{m^2+1}\right)-\frac{|ma|\sqrt{m^2+1-m^2a^2}}{m^2+1}\right]$$

which is a function of the slope m of the line l and the distance a of the point $(a,0)$ from the origin as a fraction of the overall area of the circle, which is π. This plot is shown in Figure **2**, below.

As one can see from the graph, at a distance of zero from the origin, the only possible split of the circle is in 50-50%. This is evident also analytically, since the value of the function at a distance of $a=0$ is given by

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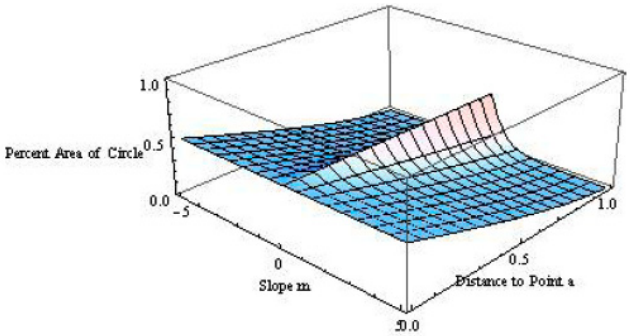

Figure 2: Plot of $A(a,m)=\cos^{-1}(\frac{|ma|}{m^2+1})-\frac{|ma|\sqrt{m^2+1-m^2a^2}}{m^2+1}$.

$$A(0,m)=\cos^{-1}(0)-\frac{0}{1}=\frac{\pi}{2}.$$

Similarly shown by the graph, as the distance a increases, a 50-50% split is always possible, but smaller regions are also created as the slope approaches infinity. The minimum (and thus the maximum as well) is attained as expected through a vertical line through (a,b) given by

$$\lim_{m\to\infty} A(a,m)=\theta-a\sqrt{1-a^2}.$$

PARALLELOGRAM

Next we will look at the case for a paralleogram, in which we will give a completely geometric proof. In Figure **3**, let ABCD be an arbitrary parallelogram and O be an arbitrary point in the interior of the parallelogram. We construct the line segments $\overline{OF}$ and $\overline{OE}$ parallel to sides $\overline{AB}$ and $\overline{AD}$, respectively. Moving at the opposite vertex C we pick the points G and H such that $CG=AF$ and $CH=AE$. The line connecting point O and O' will be the line that splits the parallelogram in 50-50%.

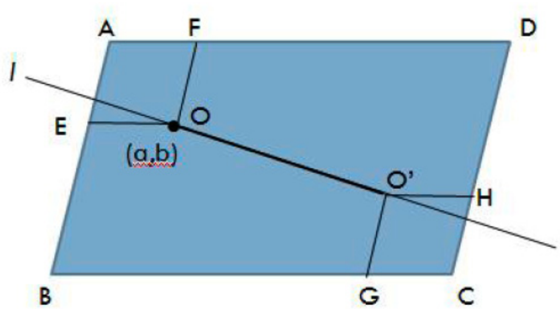

Figure 3: An arbitrary parallelogram, with vertices labeled and an arbitrary line l through the point (a,b).

SCALENE TRIANGLE

For a scalene triangle, we note first that if the point chosen lies in any of the medians then the respective median gives us the 50-50% split. To simplify the calculations we will assume that the largest side of the triangle has a length of 1 and we place that side on the x-axis with one vertex at the origin and another at $(1,0)$. Let the third vertex lie at (k,h), as depicted in Figure **4**. We will like to note that in this scenario both $h,k\in(0,1)$.

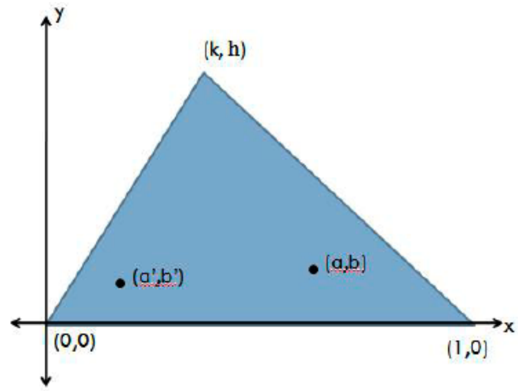

Figure 4: A scalene triangle, with vertices $(0,0)$, $(1,0)$ and (k,h).

Clearly, if (a,b) lies on any one of the medians, we will have a 50-50% split. For all of the other cases we will have a complete proof. We will assume for simplicity of the argument that $a<k$. We will see at the end of the proof that the case $a>k$ will yield a very similar result.

The idea of the proof is to show that the function that gives the area of the region inside the triangle and under the line l is a continuous function of m (the slope). This function is a piecewise defined function of m, the slope of the line through (a,b). We will have to look at the points where the function changes its behavior. If the line l is a vertical line then the area to the left of our line is the area we will be loking for and that area is equal to $A=\frac{ha^2}{2k}$. As the slope of the line l changes from $(-\infty,\frac{b}{a-1})$ the area inside our triangle and below the line l is given by

$$A(m)=\frac{1}{2}\frac{h(b-ma)}{h-mk}\frac{am-b}{m}=-\frac{h(b-ma)^2}{2m(h-mk)}$$

which is clearly a continuous function on our interval. Likewise, as $m\in(\frac{b}{a-1},\frac{b}{a})$ the area inside our triangle and below our line l is given by

$$A(m)=\frac{h}{2}-\frac{1}{2}\det\begin{bmatrix} k & h & 1 \\ \frac{k(b-ma)}{h-mk} & \frac{h(b-ma)}{h-mk} & 1 \\ \frac{(k-1)(b-ma)+h}{h-m(k-1)} & \frac{h(b-ma+m)}{h-m(k-1)} & 1 \end{bmatrix}$$
$$=\frac{h}{2}-\left[\frac{h(-b+h+ma-mk)^2}{2(h+m-km)(h-mk)}\right]$$

In the previous line we use Mathematica to compute and simplify that determinant. Continuing in the same fashion, for $m \in (\frac{b}{a}, \frac{h-b}{k-a})$ the area inside our triangle and below our line l is going to be given by

$$A(m) = \frac{(1-\frac{am-b}{m})(\frac{h(b-ma+m)}{h-m(k-1)})}{2} = \frac{h(m-ma+b)^2}{2m(h-m(k-1))}.$$

Finnaly, as $m \in (\frac{h-b}{k-a}, \infty)$ we obtain that the area inside the triangle and below our line l is given by

$$A(m) = \frac{h}{2} - (-\frac{h(b-ma)^2}{2m(h-mk)}).$$

Therefore, the function that gives us the area inside the triangle and under the line l is a piecewise defined function defined by:

$$A(m) = \begin{cases} -\frac{h(b-ma)^2}{2m(h-mk)} & for -\infty < m < \frac{b}{a-1} \\ \frac{h}{2} - \left[\frac{h(-b+h+ma-mk)^2}{2(h+m-km)(h-mk)}\right] & for \frac{b}{a-1} \le m < \frac{b}{a} \\ \frac{h(m-ma+b)^2}{2m(h-m(k-1))} & for \frac{b}{a} \le m \le \frac{h-b}{k-a} \\ \frac{h}{2} + \frac{h(b-ma)^2}{2m(h-mk)} & for \frac{h-b}{k-a} < m < \infty. \end{cases}$$

Although $A(m)$ is not the prettiest of function we have to note that for each of our four pieces the singularities of the rational functions are not in the respective intervals, which mean that each of our piece is continuous on its own interval. Therefore, we only have to check the continuity at the points where our function changes its behavior.

$$\lim_{m \to (\frac{b}{a-1})^-} -\frac{h(b-ma)^2}{2m(h-mk)} = \lim_{m \to (\frac{b}{a-1})^+} \frac{h}{2} - \left[\frac{h(-b+h+ma-mk)^2}{2(h+m-km)(h-mk)}\right] = -\frac{hb}{ha-h-bk}$$

$$\lim_{m \to (\frac{b}{a})^-} \frac{h}{2} - \left[\frac{h(-b+h+ma-mk)^2}{2(h+m-km)(h-mk)}\right] = \lim_{m \to (\frac{b}{a})^+} \frac{h(m-ma+b)^2}{2m(h-m(k-1))} = \frac{hb}{2(ha+b-bk)}$$

$$\lim_{m \to (\frac{h-b}{k-a})^-} \frac{h(m-ma+b)^2}{2m(h-m(k-1))} = \lim_{m \to (\frac{h-b}{k-a})^+} \frac{h}{2} + \frac{h(b-ma)^2}{2m(h-mk)} = \frac{h}{2} + \frac{h(bk-ha)}{2(h-b)}$$

Therefore, our function $A(m)$ is a continuous function over the real numbers and by the Intermediate Value Theorem will take any value between $\frac{ha^2}{2k}$ and $\frac{h}{2} - \frac{ha^2}{2k}$. We remark that $\frac{h}{4}$ which is half of the area of our triangle will always sit between these two values and that completes our proof.

FINAL REMARKS AND CONJECTURE

In summary, all of the shapes we have studied have shown to have a possible 50-50% split for any point (a,b). We believe that this will be true for any convex region of the plane but we do not have a proof of this fact.

REFERENCES

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