On a Piece of Paper an a Pair of Scissors

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Abstract: We discuss the problem of splitting a given geometrical plane figure in two regions of equal area by drawing a line through a given point in the plane. We also discuss the possibility of splitting the figure into regions in proportions *p*% and (1 − *p*)% of the area of the given figure.

Keywords: Lebesgue measure, continuity.

1. INTRODUCTION

The problem of splitting a given geometrical figure in two regions of equal area by drawing a line through a given point in the interior as been discussed in [1]. It is reported that it is simple for basic geometric figures and it is suggested it is true for convex figures but no proof is given. We reconsider the problem for area given by the Lebesgue measure of the set (which includes the area considered in [1]) and for any given point in the plane. We show that it is always possible to split a figure in two regions of equal area, and present bounds on the proportions of possible splitting of the figure into regions in p % and $(1-p)$ % of the area of the given figure.

2. PARTITION OF SETS

Let $\mathbb{R}^2$ be the two dimensional Euclidean space with scalar product $u \bullet v$ and norm $\|v\| = \left(v \bullet v\right)^{1/2}$. Let a be a point in $\mathbb{R}^2$, and $n(\theta) = \left(\cos(\theta), \sin(\theta)\right)$ be a unit vector in $\mathbb{R}^2$, where θ is direction angle. We consider the partition of $\mathbb{R}^2$ given by

$$\mathbb{R}^2 = H_+(a;\theta) \cup H_0(a;\theta) \cup H_-(a;\theta),$$

where

$$\begin{aligned} H_+(a;\theta) &= \left\{x \in \mathbb{R}^2 \mid (x-a) \bullet n(\theta) > 0\right\}, \\ H_0(a;\theta) &= \left\{x \in \mathbb{R}^2 \mid (x-a) \bullet n(\theta) = 0\right\}, \\ H_-(a;\theta) &= \left\{x \in \mathbb{R}^2 \mid (x-a) \bullet n(\theta) < 0\right\}. \end{aligned}$$

In fact $H_0(a;\theta)$ is a line passing through a with normal $n(\theta)$.

Let G be any subset of $\mathbb{R}^2$. We consider the following partition of G given by

$$G = G_+(\theta) \cup G_0(\theta) \cup G_-(\theta)$$

where

$$\begin{aligned} G_+(\theta) &= G \cap H_+(a;\theta), \\ G_0(\theta) &= G \cap H_0(a;\theta), \\ G_-(\theta) &= G \cap H_-(a;\theta). \end{aligned}$$

Let us remark that

$$G_+(\theta+\pi) = G_-(\theta) \quad \text{and} \quad G_+(\theta-\pi) = G_-(\theta) \tag{2.1}$$

for any $\theta \in \mathbb{R}$.

For any two angles, θ and $\tilde{\theta}$, the sets

$$H_+(a;\tilde{\theta}) \cap H_-(a;\theta) \quad \text{and} \quad H_-(a;\tilde{\theta}) \cap H_+(a;\theta) \tag{2.2}$$

are two infinite circular sectors with vertex at a, of angle $|\tilde{\theta} - \theta|$ and in opposite directions.

We consider also the closed ball centerad at a of radius R given by

$$\bar{B}(a;R) = \left\{x \in \mathbb{R}^2 \mid \left\|x-a\right\| \le R\right\}.$$

The sets

$$\begin{aligned} &\left[H_+(a;\tilde{\theta}) \cap H_-(a;\theta)\right] \cap \bar{B}(a;R) \\ &\text{and} \quad \left[H_-(a;\tilde{\theta}) \cap H_+(a;\theta)\right] \cap \bar{B}(a;R) \end{aligned} \tag{2.3}$$

are bounded sectors with vertex at a, of angle $|\tilde{\theta} - \theta|$ and radius R, and in opposite directions.

3. LEBESGUE MEASURABLE SETS

For a measurable subset G of $\mathbb{R}^2$, its area is given by its Lebesgue measure noted $\operatorname{meas}(G)$, we have

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$$\begin{aligned}\mathrm{meas}(G) &= \mathrm{meas}(G_+(\theta)) + \mathrm{meas}(G_0(\theta)) + \mathrm{meas}(G_-(\theta)) \\ &= \mathrm{meas}(G_+(\theta)) + \mathrm{meas}(G_-(\theta))\end{aligned} \tag{3.1}$$

because $\mathrm{meas}(G_0(\theta)) = 0$.

Lemma 3.1. Let G be a bounded measurable set, then $\mathrm{meas}(G_+(\theta))$ and $\mathrm{meas}(G_-(\theta))$ are continuous functions of θ.

Proof. Let $\tilde{\theta} > \theta$ such that $0 \le \tilde{\theta} - \theta \le \Delta\theta$ (the case $\tilde{\theta} < \theta$ is similar), since

$$\begin{aligned}G_+(\theta) = &\left[G_+(\theta) \cap G_+(\tilde{\theta})\right] \cup \left[G_+(\theta) \cap G_0(\tilde{\theta})\right] \\ &\cup \left[G_+(\theta) \cap G_-(\tilde{\theta})\right]\end{aligned}$$

and

$$\begin{aligned}G_+(\tilde{\theta}) = &\left[G_+(\tilde{\theta}) \cap G_+(\theta)\right] \cup \left[G_+(\tilde{\theta}) \cap G_0(\theta)\right] \\ &\cup \left[G_+(\tilde{\theta}) \cap G_-(\theta)\right]\end{aligned}$$

it follows that

$$\begin{aligned}&| \mathrm{meas}(G_+(\tilde{\theta})) - \mathrm{meas}(G_+(\theta)) | \le | \mathrm{meas}(G_+(\theta) \cap G_-(\tilde{\theta})) \\ &- \mathrm{meas}(G_+(\tilde{\theta}) \cap G_-(\theta)) | .\end{aligned}$$

Since $G_+(\theta) \cap G_-(\tilde{\theta})$ and $G_+(\tilde{\theta}) \cap G_-(\theta)$ are both in the bounded circular sectors (2.3) of angle $| \tilde{\theta} - \theta | \le \Delta\theta$ and radius, say R such that $G \in \bar{B}(a;R)$, not depending on the angles, then

$$\mathrm{meas}(G_+(\theta) \cap G_-(\tilde{\theta})) \le R^2 \frac{\Delta\theta}{2},$$

and

$$\mathrm{meas}(G_+(\tilde{\theta}) \cap G_-(\theta)) \le R^2 \frac{\Delta\theta}{2},$$

and the continuity of the functions follows.

We can extend the continuity to unbounded set using the following lemma.

Lemma 3.2. If G is an unbounded set and $\mathrm{meas}(G) < \infty$, then for any $\epsilon > 0$ there exists a sufficiently large $R(\epsilon) > 0$ such that

$$\begin{aligned}&\mathrm{meas}(G \cap \bar{B}(a;R(\epsilon))) \le \mathrm{meas}(G) \\ &\le \mathrm{meas}(G \cap \bar{B}(a;R(\epsilon))) + \epsilon\end{aligned} \tag{3.2}$$

Proof. Because $G \cap \bar{B}(a;R) \subset G$, $G \cap \bar{B}(a;R_1) \subset G \cap \bar{B}(a;R_2)$ for $R_1 < R_2$, and $\bigcup_{R>0}\left\{G \cap \bar{B}(a;R)\right\} = G$, then the properties of Lebesgue measure [2] imply that

$$meas(G \cap \bar{B}(a;R)) \le meas(G)$$

and

$$\lim_{R\to\infty} meas(G \cap \bar{B}(a;R)) = meas(G)$$

and the result follows.

Hence we have proved the following result.

Theorem 3.3. Let G be a measurable set. If $\mathrm{meas}(G) < \infty$ then $\mathrm{meas}(G_+(\theta))$ and $\mathrm{meas}(G_-(\theta))$ are continuous functions of θ.

4. CONSEQUENCES OF CONTINUITY

The two continuous functions $\mathrm{meas}(G_+(\theta))$ and $\mathrm{meas}(G_-(\theta))$ both reach their maximal and minimal values on the interval $\left[0, 2\pi\right]$. Let

$$\begin{aligned}M_+ &= \max\left\{\mathrm{meas}(G_+(\theta)) \mid \theta \in \left[0,\pi\right]\right\}, \\ m_+ &= \min\left\{\mathrm{meas}(G_+(\theta)) \mid \theta \in \left[0,\pi\right]\right\}, \\ M_- &= \max\left\{\mathrm{meas}(G_-(\theta)) \mid \theta \in \left[0,\pi\right]\right\}, \\ m_- &= \min\left\{\mathrm{meas}(G_-(\theta)) \mid \theta \in \left[0,\pi\right]\right\}.\end{aligned}$$

Since (2.1) holds then $M_+ = M_-(= M)$ and $m_+ = m_-(= m)$. Also, from (3.1), $\mathrm{meas}(G_+(\theta))$ have its maximum, respectively minimum, values where $\mathrm{meas}(G_-(\theta))$ has its minimum, respectively maximum, values. Then $M + m = \mathrm{meas}(G)$.

The set of splitting proportions is $\left[\frac{m}{\mathrm{meas}(G)}\%, \frac{M}{\mathrm{meas}(G)}\%\right]$, and by continuity and the Intermediate Value Theorem any proportion in this interval is realizable. In particular, since $m \le \frac{1}{2}\mathrm{meas}(G) \le M$ it is always possible to split G in two equal area regions.

5. CONCLUSION

We have extended results and proved a conjecture stated in [1]. Extension to $\mathbb{R}^n$ for $n > 2$ could be considered.

REFERENCES

[1] Mihai C, Rakowski C. A piece of paper and a pair of scissors. Journal of Basic & Applied Sciences 2014; 10: 428-30. http://dx.doi.org/10.6000/1927-5129.2014.10.56

[2] Halmos PR. Measure Theory. New York: Springer-Verlag 1974.

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