Exceptional Sets for Subharmonic Functions

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Abstract: Blanchet has shown that hypersurfaces of class C^1 are removable singularities for subharmonic functions, provided the considered subharmonic functions satisfy certain assumptions. Later we showed that, in certain cases, it is sufficient that the exceptional sets are of finite (n-1)-dimensional Hausdorff measure. Now we improve our results still further, relaxing our previous assumptions imposed on the considered subharmonic functions.

Keywords: Subharmonic, Hausdorff measure, exceptional sets.

1. INTRODUCTION

1.1. Previous Results

Blanchet [1], Theorems 3.1, 3.2 and 3.3, p. 312-3, gave the following removability results.

Blanchet's theorem. Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let S be a hypersurface of class $\mathcal{C}^1$ which divides Ω into two subdomains Ω_1 and Ω_2. Let $u \in \mathcal{C}^0(\Omega) \cap \mathcal{C}^2(\Omega_1 \cup \Omega_2)$ be subharmonic (respectively convex (or respectively plurisubharmonic provided Ω is then a domain in $\mathbb{C}^n$, $n \geq 1$)) in Ω_1 and Ω_2. If $u_i = u \mid \Omega_i \in \mathcal{C}^1(\Omega_i \cup S)$, $i = 1, 2$, and

$$\frac{\partial u_i}{\partial \overset{-k}{n}} \geq \frac{\partial u_k}{\partial \overset{-k}{n}} \qquad (1)$$

on S with $i, k = 1, 2$, then u is subharmonic (respectively convex (or respectively plurisubharmonic)) in Ω.

Above $\overset{-k}{n} = (\overset{-k}{n_1}, \ldots, \overset{-k}{n_n})$ is the unit normal exterior to Ω_k, and $u_k \in \mathcal{C}^1(\Omega_k \cup S)$, $k = 1, 2$, means that there exist n functions v_k^j, $j = 1, \ldots, n$, continuous on $\Omega_k \cup S$, such that

$$v_k^j(x) = \frac{\partial u_k}{\partial x_j}(x)$$

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for all $x \in \Omega_k$, $k = 1, 2$ and $j = 1, \ldots, n$.

The following example shows that one cannot drop the above condition (1) in Blanchet's theorem.

Example. The function $u : \mathbb{R}^2 \to \mathbb{R}$,

$$u(z) = u(x + iy) = u(x, y) := \begin{cases} 1 + x, & \text{when } x < 0, \\ 1 - x, & \text{when } x \geq 0, \end{cases}$$

is continuous in $\mathbb{R}^2$ and subharmonic, even harmonic in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. It is easy to see that u does not satisfy the condition (1) on $S = \{0\} \times \mathbb{R}$ and that u is not subharmonic in $\mathbb{R}^2$.

1.2. Already in [2], Theorem 4, p. 181-2, we have given partial improvements to the cited subharmonic removability results of Blanchet. Now we improve our previous improvements still further, see Theorem and Corollaries 1 and 2 below. Instead of hypersurfaces of class $\mathcal{C}^1$, we will below allow arbitrary sets of finite (*n*-1)-dimensional Hausdorff measure as exceptional sets. Then we must, however, replace the condition (1) by another, related condition, the condition (iv) in our Theorem. Moreover, we must also impose an additional integrability condition on the second partial derivatives $\frac{\partial^2 u}{\partial x_j^2}$, $j = 1, \ldots, n$.

Our method of proof is rather elementary, thus natural, with the only exception that we need one geometric measure theory result of Federer.

1.3. Notation

Our notation is more or less standard, see [2, 3]. However, for the convenience of the reader we recall here the following. We use the common convention $0 \cdot \pm \infty = 0$. In integrals we will write dx for the

Lebesgue measure in $\mathbb{R}^k$, $k \in \mathbb{N}$. Let $0 \le \alpha \le n$ and $A \subset \mathbb{R}^n$, $n \ge 1$. Then we write $\mathcal{H}^\alpha(A)$ for the α-dimensional Hausdorff (outer) measure of A. Recall that $\mathcal{H}^0(A)$ is the number of points of A. $\mathcal{L}^p_{\mathtt{loc}}(\Omega)$, $p > 0$, is the space of functions u in Ω for which $|u|^p$ is locally integrable on Ω. If $x = (x_1,...,x_n) \in \mathbb{R}^n$, $n \ge 2$, and $j \in \mathbb{N}$, $1 \le j \le n$, then we write $x = (x_j, X_j)$, where $X_j = (x_1,...,x_{j-1}, x_{j+1},...,x_n)$. Moreover, if $A \subset \mathbb{R}^n$, $1 \le j \le n$, and $x_j^0 \in \mathbb{R}$, $X_j^0 \in \mathbb{R}^{n-1}$, we write.

$$A(x_j^0) = \{ X_j \in \mathbb{R}^{n-1} \ : \ x = (x_j^0, X_j) \in A \},$$
$$A(X_j^0) = \{ x_j \in \mathbb{R} \ : \ x = (x_j, X_j^0) \in A \}.$$

For the definition and properties of subharmonic functions, see e.g. [4-7].

2. AN EXCEPTIONAL SET FOR SUBHARMONIC FUNCTIONS

2.1. On the Extension of Subharmonic Functions

The following measure theoretic result is essential for our proof:

Lemma. ([8], Theorem 2.10.25, p. 188) Suppose that $A \subset \mathbb{R}^n$ is such that $\mathcal{H}^{n-1}(A) < +\infty$. Then for all j, $1 \le j \le n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ the set $A(X_j)$ is finite.

Our result is:

Theorem. Suppose that Ω is a domain in $\mathbb{R}^n$, $n \ge 2$. Let $E \subset \Omega$ be closed in Ω and $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \to [-\infty, +\infty]$ be such that the following conditions are satisfied:

(i) $u \in \mathcal{L}^1_{\mathtt{loc}}(\Omega)$.

(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$.

(iii) For each $j, 1 \le j \le n, \dfrac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathtt{loc}}(\Omega)$.

(iv) For each $j, 1 \le j \le n,$ and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds:

For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j), l = 1, 2,...,$ such that

(iv(a)) $x_{j,l}^{0,1} \nearrow x_j^0, x_{j,l}^{0,2} \searrow x_j^0$, and

$$\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R},$$

(iv(b)) $-\infty < \lim_{l \to +\infty} \dfrac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \le$

$$\lim_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$$

(v) u is subharmonic in $\Omega \setminus E$.

Then $u \big| \Omega \setminus E$ has a subharmonic extension to Ω.

Proof. It is sufficient to show that

$$\int u(x) \Delta \varphi(x) dx \ge 0$$

for all nonnegative testfunctions $\varphi \in \mathcal{D}(\Omega)$. Take $\varphi \in \mathcal{D}(\Omega)$, $\varphi \ge 0$, arbitrarily. Let $K = \mathtt{spt}\, \varphi$. Choose a domain Ω_1 such that $K \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ and $\overline{\Omega_1}$ is compact. Since $u \in \mathcal{C}^2(\Omega \setminus E)$ and u is subharmonic in $\Omega \setminus E, \Delta u(x) \ge 0$ for all $x \in \Omega \setminus E$. Thus the claim follows if we show that

$$\int u(x) \Delta \varphi(x) dx \ge \int \Delta u(x) \varphi(x) dx.$$

For this purpose fix j, $1 \le j \le n$, arbitrarily for a while. By Fubini's theorem, see e.g. [9], Theorem 7.8, p. 150,

$$\int u(x) \frac{\partial^2 \varphi}{\partial x_j^2}(x) dx = \int \Big[\int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j) dx_j \Big] dX_j.$$

Using Lemma, assumptions (i), (ii) and (iii), and Fubini's theorem, we see that for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$,

$$\begin{cases} u(\cdot\ , X_j) \in \mathcal{L}^1_{\mathtt{loc}}(\Omega(X_j)), \\ \dfrac{\partial^2 u}{\partial x_j^2}(\ \cdot\ , X_j) \in \mathcal{L}^1_{\mathtt{loc}}(\Omega(X_j)), \\ E(X_j) \text{ is finite, thus there exists} \\ M = M(X_j) \in \mathbb{N} \text{ such that} \\ E(X_j) = \{x_j^1,...,x_j^M\} \text{ where } x_j^k < x_j^{k+1}, k = 1,...,M-1. \end{cases} \tag{2}$$

Let $X_j \in \mathbb{R}^{n-1}$ be arbitrary as above in (2). We may suppose that $\Omega(X_j)$ is a finite interval. Choose for each $k = 1,\dots,M$ numbers $a_k, b_k \in (\Omega \setminus E)(X_j)$ such that $a_k < x_j^k < b_k$, $k = 1,\dots,M$, $a_{k+1} = b_k$, $k = 1,\dots,M-1$, and that $a_1, b_M \in (\Omega \setminus \overline{\Omega_1})(X_j)$.

With the aid of (iv) we find for each $x_j^k \in E(X_j)$ sequences $x_{j,l}^{k,1}, x_{j,l}^{k,2} \in (\Omega \setminus E)(X_j)$, $l = 1,2,\dots,$ for which

(a) $x_{j,l}^{k,1} \nearrow x_j^k, x_{j,l}^{k,2} \searrow x_j^k$ and

$$\lim_{l \to +\infty} u(x_{j,l}^{k,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{k,2}, X_j) \in \mathbb{R},$$

(b) $-\infty < \lim_{l \to +\infty} \dfrac{\partial u}{\partial x_j}(x_{j,l}^{k,1}, X_j) \le$

$$\le \lim_{l \to +\infty} \dfrac{\partial u}{\partial x_j}(x_{j,l}^{k,2}, X_j) < +\infty$$

Take k, $1 \le k \le M$, arbitrarily and consider the interval (a_k, b_k), where $a_k < x_j^k < b_k$. To simplify the notation, write $a := a_k$, $b := b_k$ and $x_j^0 := x_j^k$. Then

$$a < x_{j,l}^{0,1} \nearrow x_j^0, \ b > x_{j,l}^{0,2} \searrow x_j^0 \text{ as } l \to +\infty.$$

Then just partial integration!

$$\int_a^b u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j = \int_a^{x_j^0} u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j +$$

$$+ \int_{x_j^0}^b u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j$$

$$= \lim_{l \to +\infty}\int_a^{x_{j,l}^{0,1}} u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j +$$

$$+ \lim_{l \to +\infty}\int_{x_{j,l}^{0,2}}^b u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j$$

$$= \lim_{l \to +\infty}\left[\left[u(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\right]_a^{x_{j,l}^{0,1}} - \int_a^{x_{j,l}^{0,1}}\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j\right] +$$

$$+ \lim_{l \to +\infty}\left[\left[u(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\right]_{x_{j,l}^{0,2}}^b - \int_{x_{j,l}^{0,2}}^b\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j\right]$$

$$= \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$+ \lim_{l \to +\infty}\left[u(x_{j,l}^{0,1}, X_j)\frac{\partial \varphi}{\partial x_j}(x_{j,l}^{0,1}, X_j) - \int_a^{x_{j,l}^{0,1}}\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j\right] +$$

$$- \lim_{l \to +\infty}\left[u(x_{j,l}^{0,2}, X_j)\frac{\partial \varphi}{\partial x_j}(x_{j,l}^{0,2}, X_j) + \int_{x_{j,l}^{0,2}}^b\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j\right]$$

$$= \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$- \lim_{l \to +\infty}\int_a^{x_{j,l}^{0,1}}\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j +$$

$$- \lim_{l \to +\infty}\int_{x_{j,l}^{0,2}}^b\frac{\partial u}{\partial x_j}(x_j, X_j)\frac{\partial \varphi}{\partial x_j}(x_j, X_j)\,dx_j$$

$$= \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$- \lim_{l \to +\infty}\left[\left[\frac{\partial u}{\partial x_j}(x_j, X_j)\varphi(x_j, X_j)\right]_a^{x_{j,l}^{0,1}} - \int_a^{x_{j,l}^{0,1}}\frac{\partial^2 u}{\partial x_j^2}(x_j, X_j)\,\varphi(x_j, X_j)\,dx_j\right] +$$

$$- \lim_{l \to +\infty}\left[\left[\frac{\partial u}{\partial x_j}(x_j, X_j)\varphi(x_j, X_j)\right]_{x_{j,l}^{0,2}}^b - \int_{x_{j,l}^{0,2}}^b\frac{\partial^2 u}{\partial x_j^2}(x_j, X_j)\,\varphi(x_j, X_j)\,dx_j\right]$$

$$= \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$+ \left[\frac{\partial u}{\partial x_j}(a, X_j)\varphi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j)\varphi(b, X_j)\right] +$$

$$- \lim_{l \to +\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j)\varphi(x_{j,l}^{0,1}, X_j)\right] + \lim_{l \to +\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j)\varphi(x_{j,l}^{0,2}, X_j)\right] +$$

$$+ \int_a^b\frac{\partial^2 u}{\partial x_j^2}(x_j, X_j)\,\varphi(x_j, X_j)\,dx_j$$

$$= \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$+ \left[\frac{\partial u}{\partial x_j}(a, X_j)\varphi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j)\varphi(b, X_j)\right] +$$

$$+ \left[\lim_{l \to +\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) - \lim_{l \to +\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j)\right]\varphi(x_j^0, X_j) +$$

$$+ \int_a^b\frac{\partial^2 u}{\partial x_j^2}(x_j, X_j)\,\varphi(x_j, X_j)\,dx_j$$

$$\ge \left[u(b, X_j)\frac{\partial \varphi}{\partial x_j}(b, X_j) - u(a, X_j)\frac{\partial \varphi}{\partial x_j}(a, X_j)\right] +$$

$$+ \left[\frac{\partial u}{\partial x_j}(a, X_j)\varphi(a, X_j) - \frac{\partial u}{\partial x_j}(b, X_j)\varphi(b, X_j)\right]$$

$$+ \int_a^b\frac{\partial^2 u}{\partial x_j^2}(x_j, X_j)\,\varphi(x_j, X_j)\,dx_j.$$

Above we have used just standard properties of limits and our assumption (iv(b)). Observe here, for example, that already the assumptions (i), (ii), (iii) and (iv(a)) imply the existence of the limits

$$\lim_{l \to +\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j)\varphi(x_{j,l}^{0,1}, X_j)\right] \text{ and}$$

$$\lim_{l \to +\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j)\varphi(x_{j,l}^{0,2}, X_j)\right].$$

To return to the original notation, we have obtained for each $k = 1,\dots,M$,

$$\int_{a_k}^{b_k} u(x_j, X_j)\frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\,dx_j \ge$$

$$\left[u(b_k, X_j)\frac{\partial \varphi}{\partial x_j}(b_k, X_j) - u(a_k, X_j)\frac{\partial \varphi}{\partial x_j}(a_k, X_j)\right] +$$

$$+\left[\frac{\partial u}{\partial x_j}(a_k,X_j)\varphi(a_k,X_j)-\frac{\partial u}{\partial x_j}(b_k,X_j)\varphi(b_k,X_j)\right]$$

$$+\int_{a_k}^{b_k}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j.$$

What remains, is just to sum over k:

$$\int u(x_j,X_j)\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j=\int_{a_1}^{b_M}u(x_j,X_j)\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j=$$

$$=\sum_{k=1}^{M}\int_{a_k}^{b_k}u(x_j,X_j)\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j\geq$$

$$\geq\sum_{k=1}^{M}\left[u(b_k,X_j)\frac{\partial\varphi}{\partial x_j}(b_k,X_j)-u(a_k,X_j)\frac{\partial\varphi}{\partial x_j}(a_k,X_j)\right]+$$

$$+\sum_{k=1}^{M}\left[\frac{\partial u}{\partial x_j}(a_k,X_j)\varphi(a_k,X_j)-\frac{\partial u}{\partial x_j}(b_k,X_j)\varphi(b_k,X_j)\right]+$$

$$+\sum_{k=1}^{M}\int_{a_k}^{b_k}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j=$$

$$=\int_{a_1}^{b_M}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j=\int\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j.$$

Above we have used the choice of the numbers $a_k,b_k,$ $k=1,\ldots,M$, and the fact that $a_1,b_M\in(\Omega\setminus\overline{\Omega_1})(X_j)$.

Integrate then with respect to X_j and use again the cited Fubini's theorem:

$$\int u(x)\frac{\partial^2\varphi}{\partial x_j^2}(x)\,dx=\int\left[\int u(x_j,X_j)\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j\right]dX_j\geq$$

$$\geq\int\left[\int\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j\right]dX_j=\int\frac{\partial^2 u}{\partial x_j^2}(x)\varphi(x)\,dx.$$

Summing over $j=1,\ldots,n$ gives the desired inequality

$$\int u(x)\Delta\varphi(x)\,dx=\int u(x)\sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial x_j^2}(x)\,dx\geq$$

$$\geq\int\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}(x)\varphi(x)\,dx=\int\Delta u(x)\varphi(x)\,dx\geq 0,$$

concluding the proof.

Corollary 1. ([2], Theorem 4, p. 181-2) Suppose that Ω is a domain in $\mathbb{R}^n$, $n\geq 2$. Let $E\subset\Omega$ be closed in Ω and $\mathcal{H}^{n-1}(E)<+\infty$. Let $u:\Omega\to\mathbb{R}$ be such that

(i) $u\in C^o(\Omega)$,

(ii) $u\in C^2(\Omega\setminus E)$,

(iii) For each $j,1\leq j\leq n,\dfrac{\partial^2 u}{\partial x_j^2}\in\mathcal{L}^1_{\mathbf{loc}}(\Omega)$.

(iv) For each $j,1\leq j\leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j\in\mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, one has

$$-\infty<\liminf_{\varepsilon\to 0+0}\frac{\partial u}{\partial x_j}(x_j^0-\varepsilon,X_j)\leq$$

$$\leq\limsup_{\varepsilon\to 0+0}\frac{\partial u}{\partial x_j}(x_j^0+\varepsilon,X_j)<+\infty\text{ for each }x_j^0\in E(X_j),$$

(v) u is subharmonic in $\Omega\setminus E$.

Then u is subharmonic.

Corollary 2. ([2], Corollary 4A, p. 185-6) *Suppose that Ω is a domain in $\mathbb{R}^n$, $n\geq 2$. Let $E\subset\Omega$ be closed in Ω and $\mathcal{H}^{n-1}(E)<+\infty$. Let $\Omega\to\mathbb{R}$ be such that*

(i) $u\in C^1(\Omega)$,

(ii) $u\in C^2(\Omega\setminus E)$,

(iii) For each $j,1\leq j\leq n,\dfrac{\partial^2 u}{\partial x_j^2}\in\mathcal{L}^1_{\mathbf{loc}}(\Omega)$,

(iv) u is subharmonic in $\Omega\setminus E$.

Then u is subharmonic.

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