Generalized Higher Order (ϕ, η, ω, π, ρ, θ, m)-Invexities in Parametric Optimality Conditions for Discrete Minmax Fractional Programming

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Abstract: First several new classes of higher order (ϕ, η, ω, π, ρ, θ, m)-invexities are introduced, and then a set of higher-order parametric necessary optimality conditions and several sets of higher order sufficient optimality conditions for a discrete minmax fractional programming problem applying various higher order (ϕ, η, ω, π, ρ, θ, m)-invexity constraints are established. The obtained results are new and generalize a wide range of results in the literature.

Keywords: Discrete minmax fractional programming, (ϕ, η, ω, π, ρ, θ, m)-invex functions, necessary optimality conditions, sufficient optimality conditions.

1. INTRODUCTION

In this communication, first several new classes of generalized second-order (ϕ, η, ω, π, ρ, θ, m)-invex functions are introduced, and then these are applied to establish a set of second-order necessary optimality conditions leading to several sets of second-order sufficient optimality conditions and theorems for the following discrete minmax fractional programming problem:

$$(P) \qquad Minimize \ \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)}$$

subject to $G_j(x) \le 0, j \in \underline{q}, H_k(x) = 0, k \in \underline{r}, x \in X$, where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i, g_i, i \in \underline{p} = \{1,2,\ldots,p\}, G_j, j \in \underline{q}$, and $H_k, k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p}, g_i(x) > 0$ for all x satisfying the constraints of (P).

The first part of this presentation deals with several new notions of the generalized second order (ϕ, η, ω, π, ρ, θ, m)-invexities, which generalize/unify most of the existing generalized invexities and variants in the literature. Then some second-order optimality conditions for our principal problem (P) are established. The obtained results can be generalized to its semiinfinite counterparts as well. Furthermore, our results can be applied to the new notion (developed in Chinchuluun and Pardalos [1], Pitea and Postalache [2-4]) of multitime multiobjective variational problems. Zalmai [13-15] introduced and investigated some significant results in a series of publications, while the results of Verma and Zalmai [11] and Verma [9] are significant to our problem on hand. For more details to this context, we refer the reader [5-16]. The results thus obtained here in this communication are new and application-oriented to context of results available in the literature.

2. PRELIMINARIES

Verma and Zalmai [11] introduced the notion of the generalized (ϕ, η, ρ, θ, m)-invexities, and further applied to establish a class of second order parametric necessary optimality conditions as well as sufficient optimality conditions for a discrete minmax fractional programming problem using the general frameworks for the (ϕ, η, ρ, θ, m)-invexities. In this section, we first generalize the notion of the generalized (ϕ, η, ρ, θ, m)-invexities, and then recall some important auxiliary results for the problem (P) on hand.

Definition 2.1. Let f be a differentiable real-valued function defined on $\mathbb{R}^n$. Then f is said to be η-invex (invex with respect to η) at y if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$f(x) - f(y) \ge \langle \nabla f(y), \eta(x,y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle a,b \rangle$ denotes the inner product of the vectors a and b; f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$.

Let f be a twice differentiable real-valued function defined on $\mathbb{R}^n$. Now we introduce the new classes of generalized second-order hybrid invex functions which

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seem to be application-oriented to developing a new optimality-duality theory for nonlinear programming based on second-order necessary and sufficient optimality conditions. We shall abbreviate "second-order invex" as *sonvex*. Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 2.2. The function f is said to be (strictly) (ϕ, η, ω, π, ρ, θ, m)-sonvex at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}, \rho : X \times X \to \mathbb{R}$, and $\eta, \omega, \pi, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X \ (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), \omega(x,x^*)\rangle)(>) \geq \langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x,x^*)\rangle + \frac{1}{2}\langle \nabla f(x^*), \pi(x,x^*)\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^m.$$

Definition 2.3. The function f is said to be (strictly) (ϕ, η, ω, π, ρ, θ, m)-pseudosonvex at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}, \rho : X \times X \to \mathbb{R}$, and $\eta, \omega, \pi, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X \ (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x,x^*)\rangle + \frac{1}{2}\langle \nabla f(x^*), \pi(x,x^*)\rangle \geq -\rho(x,x^*)\|\theta(x,x^*)\|^m$$

$$\Rightarrow \phi(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), \omega(x,x^*)\rangle)(>) \geq 0.$$

Definition 2.4. The function f is said to be (prestrictly) (ϕ, η, ω, π, ρ, θ, m)-quasisonvex at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}, \rho : X \times X \to \mathbb{R}$, and $\eta, \omega, \pi, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*) + \frac{1}{2}\langle \nabla f(x^*), \omega(x,x^*)\rangle)(<) \leq 0 \Rightarrow$$

$$\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \eta(x,x^*)\rangle + \frac{1}{2}\langle \nabla f(x^*), \pi(x,x^*)\rangle \leq -\rho(x,x^*)\|\theta(x,x^*)\|^m.$$

Here we present some examples for our new notions of generalized invex functions.

Example 1. The function f is said to be (prestrictly) (ϕ, η, π, ρ, θ, m)-quasisonvex at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, and $\eta, \pi, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*))(<) \leq 0 \Rightarrow$$

$$\frac{1}{2}\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x,x^*)\rangle + \frac{1}{2}\langle \nabla f(x^*), \pi(x,x^*)\rangle \leq -\rho(x,x^*)\|\theta(x,x^*)\|^m.$$

Example 2. The function f is said to be (prestrictly) (ϕ, π, ρ, θ, m)-quasisonvex at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, and $\pi, \theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*))(<) \leq 0 \Rightarrow$$

$$\frac{1}{2}\langle \nabla f(x^*) + \nabla^2 f(x^*)z, z\rangle + \frac{1}{2}\langle \nabla f(x^*), \pi(x,x^*)\rangle \leq -\rho(x,x^*)\|\theta(x,x^*)\|^m.$$

We recall the following results on the second order optimality conditions to the context of the main results to be established in the next section.

Theorem 2.1. [11] Let x^* be an optimal solution of (P), let $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$, and assume that the functions $f_i, g_i, i \in \underline{p}, G_j, j \in \underline{q}$, and $H_k, k \in \underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each critical direction z^*, there exist $u^* \in U, v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{2.1}$$

$$\langle z^*, \{\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*)\} z^*\rangle \geq 0, \tag{2.2}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ i \in \underline{p}, \tag{2.3}$$

$$v_j^* G_j(x^*) = 0, \ j \in \underline{q}. \tag{2.4}$$

3. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we present several second-order sufficiency results in which various generalized (ϕ, η, ω, π, ρ, θ, m)-sonvexity assumptions are imposed on the individual as well as certain combinations of the problem functions.

For the sake of the compactness, we shall use the following notations during the statements as well as the proofs of sufficiency theorems:

$$C(x,v)=\sum_{j=1}^{q}v_jG_j(x),$$

$$D_k(x,w)=w_kH_k(x),$$

$$D(x,w)=\sum_{k=1}^{r}w_kH_k(x),$$

$$\varepsilon_i(x,\lambda)=f_i(x)-\lambda g_i(x),$$

$$\varepsilon(x,u,\lambda)=\sum_{i=1}^{p}u_i[f_i(x)-\lambda g_i(x)],$$

$$g(x,v,w)=\sum_{j=1}^{q}v_jG_j(x)+\sum_{k=1}^{r}w_kH_k(x),$$

$$I_+(u)=\{i\in \underline{p}:u_i>0\},\quad J_+(v)=\{j\in \underline{q}:v_j>0\},$$
$$K_*(w)=\{k\in \underline{r}:w_k\neq 0\}.$$

During the course of proofs for our sufficiency theorems, we shall use the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 3.1. [11] For each $x\in X$,

$$\varphi(x)=\max_{1\le i\le p}\frac{f_i(x)}{g_i(x)}=\max_{u\in U}\frac{\sum_{i=1}^{p}u_if_i(x)}{\sum_{i=1}^{p}u_ig_i(x)}.$$

Theorem 3.1. Let $x^*\in\mathbb{F}$, $\lambda^*=\varphi(x^*)\ge 0$, the functions $f_i,g_i,i\in\underline{p},G_j,j\in\underline{q}$, and $H_k,k\in\underline{r}$, be twice differentiable at x^*. Assume that for each critical direction z^*, there exist $u^*\in U, v^*\in\mathbb{R}^q_+$, and $w^*\in\mathbb{R}^r$ such that

$$\sum_{i=1}^{p}u_i^*[\nabla f_i(x^*)-\lambda^*\nabla g_i(x^*)]+\sum_{j=1}^{q}v_j^*\nabla G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla H_k(x^*)=0, \tag{3.1}$$

$$\langle z^*,\{\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)-\lambda^*\nabla^2 g_i(x^*)]+\sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)\}z^*\rangle\ge 0, \tag{3.2}$$

$$u_i^*[f_i(x^*)-\lambda^*g_i(x^*)]-\frac{1}{2}\langle\omega(x,x^*),\nabla f_i(x^*)-\lambda\nabla g_i(x^*)\rangle\ge 0,\ i\in\underline{p}, \tag{3.3}$$

$$v_j^*G_j(x^*)-\frac{1}{2}\langle\omega(x,x^*),v_j^*\nabla G_j(x^*)\rangle\ge 0,\ j\in\underline{q}, \tag{3.4}$$

$$w_k^*H_k(x^*)-\frac{1}{2}\langle\omega(x,x^*),w_k^*\nabla H_k(x^*)\rangle\ge 0,\ k\in\underline{r}, \tag{3.5}$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), defined by

$$\mathbb{F}=\{x\in X:G_j(x)\le 0,j\in\underline{q},H_k(x)=0,k\in\underline{r}\}.$$

In addition, assume that any one of the following six sets of conditions holds:

(a) (i) for each $i\in I_+\equiv I_+(u^*), f_i$ is (ϕ, η, ω, π, $\bar{\rho}_i$, θ,m)-sonvex and $-g_i$ is (ϕ, η, ω, π, $\tilde{\rho}_i$, θ, m)-sonvex at x^*,ϕ is superlinear, and $\phi(a)\ge 0\Rightarrow a\ge 0$;

(ii) for each $j\in J_+\equiv J_+(v^*),G_j$ is ($\hat{\phi}_j$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*,\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;

(iii) for each $k\in K_*\equiv K_*(w^*),\xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^* and $\breve{\phi}_k(0)=0$;

(iv) $\rho^*(x,x^*)+\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)\ge 0$ for all $x\in\mathbb{F}$, where $\rho^*(x,x^*)=\sum_{i\in I_+}u_i^*[\bar{\rho}_i(x,x^*)+\lambda^*\tilde{\rho}_i(x,x^*)]$;

(b) (i) for each $i\in I_+, f_i$ is (ϕ, η, ω, π, $\bar{\rho}_i$, θ, m)-sonvex and $-g_i$ is (ϕ, η, ω, π, $\tilde{\rho}_i$, θ, m)-sonvex at x^*,ϕ is superlinear, and $\phi(a)\ge 0\Rightarrow a\ge 0$;

(ii) $C(\cdot,v^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*,\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) for each $k\in K_*,\xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^* and $\breve{\phi}_k(0)=0$;

(iv) $\rho^*(x,x^*)+\hat{\rho}(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)\ge 0$ for all $x\in\mathbb{F}$;

(c) (i) for each $i \in I_+, f_i$ is (ϕ, ω, η, π, $\bar{\rho}_i$, θ, m)-sonvex and $-g_i$ is (ϕ, η, ω, π, $\tilde{\rho}_i$, θ, m)-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+, G_j$ is ($\hat{\phi}_j$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*, \hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;

(iii) $\xi \to D(\xi, w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonvex at x^* and $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+, f_i$ is (ϕ, η, ω, π, $\bar{\rho}_i$, θ, m)-sonvex and $-g_i$ is hybrid (ϕ, η, ω, π, $\tilde{\rho}_i$, θ, m)-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to C(\xi, v^*)$ is ($\hat{\phi}$, ω, η, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*, \hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) $\xi \to D(\xi, w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonvex at x^* and $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+, f_i$ is (ϕ, η, ω, π, $\bar{\rho}_i$, θ, m)-sonvex and $-g_i$ is (ϕ, η, ω, π, $\tilde{\rho}_i$, θ, m)-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to g(\xi, v^*, w^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*, \hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) the Lagrangian-type function

$$\xi \to L(\xi, u^*, v^*, w^*, \lambda^*) = \sum_{i=1}^{p} u_i^* [f_i(\xi) - \lambda^* g_i(\xi)] + \sum_{j=1}^{q} v_j^* G_j(\xi) + \sum_{k=1}^{r} w_k^* H_k(\xi)$$

is (ϕ, η, ω, π, ρ, θ, m)-pseudosonvex at x^*, $\rho(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, $\phi(a) \geq 0 \Rightarrow a \geq 0$, and

$$(L(x^*, u^*, v^*, w^*, \lambda^*) - \frac{1}{2}\langle \nabla L(x^*, u^*, v^*, w^*, \lambda^*), \omega(x, x^*) \rangle) \geq 0.$$

Then x^* is an optimal solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a): Using the hypotheses specified in (i), we have for each $i \in I_+$,

$$\phi(f_i(x) - f_i(x^*) + \frac{1}{2}\langle \nabla f_i(x^*), \omega(x, x^*) \rangle)$$

$$\geq \langle \nabla f_i(x^*) + \frac{1}{2}\nabla^2 f_i(x^*) z^*, \eta(x, x^*) \rangle + \frac{1}{2}\langle \nabla f_i(x^*), \pi(x, x^*) \rangle$$

$$+ \bar{\rho}_i(x, x^*) \|\theta(x, x^*)\|^m$$

and

$$\phi(-g_i(x) + g_i(x^*) - \frac{1}{2}\langle \nabla g_i(x^*), \omega(x, x^*) \rangle)$$

$$\geq -\langle \nabla g_i(x^*) + \frac{1}{2}\nabla^2 g_i(x^*) z^*, \eta(x, x^*) \rangle - \frac{1}{2}\langle \nabla g_i(x^*), \pi(x, x^*) \rangle$$

$$+ \tilde{\rho}_i(x, x^*) \|\theta(x, x^*)\|^m.$$

As $\lambda^* \geq 0, u^* \geq 0, \sum_{i=1}^{p} u_i^* = 1$, and ϕ is superlinear, we deduce from the above inequalities that

$$\phi(\sum_{i=1}^{p} u_i^* [f_i(x) - \lambda^* g_i(x)] - \sum_{i=1}^{p} u_i^* [f_i(x^*) - \lambda^* g_i(x^*)]$$

$$+ \frac{1}{2}\langle \sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \omega(x, x^*) \rangle)$$

$$\geq \frac{1}{2}\langle \sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] z^*, \eta(x, x^*) \rangle$$

$$+ \langle \sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \eta(x, x^*) \rangle$$

$$+ \frac{1}{2}\langle \sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \pi(x, x^*) \rangle$$

$$+ \sum_{i \in I_+} u_i^* [\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)] \|\theta(x, x^*)\|^m. \quad (3.6)$$

Since $x \in \mathbb{F}$ and (3.4) holds, it follows from the properties of the functions $\hat{\phi}_j$ that for each $j \in J_+$, $(v_j^* G_j(x) \leq 0 \leq v_j^* G_j(x^*) - \frac{1}{2}\langle \omega(x, x^*), v_j^* \nabla G_j(x^*) \rangle$, which implies

$$\hat{\phi}_j(v_j^* G_j(x) - v_j^* G_j(x^*) + \frac{1}{2}\langle \omega(x, x^*), v_j^* \nabla G_j(x^*) \rangle) \leq 0$$

which in view of (ii) implies that

$$\langle \nabla G_j(x^*) + \frac{1}{2}\nabla^2 G_j(x^*)z^*, \eta(x,x^*)\rangle + \frac{1}{2}\langle \nabla G_j(x^*), \pi(x,x^*)\rangle \leq -\hat{\rho}_j(x,x^*)\left\|\theta(x,x^*)\right\|^m.$$

As $v_j^* \geq 0$ for each $j \in \underline{q}$ and $v_j^* = 0$ for each $j \in \underline{q} \setminus J_+$ (complement of J_+ relative to $\underline{q}$), the above inequalities yield

$$\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*)z^*, \eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*), \pi(x,x^*)\rangle$$

$$\leq -\sum_{j\in J_+} v_j^* \hat{\rho}_j(x,x^*)\left\|\theta(x,x^*)\right\|^m. \qquad (3.7)$$

In a similar manner, we can show that (iii) leads to the following inequality:

$$\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) + \frac{1}{2}\sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*)z^*, \eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*), \pi(x,x^*)\rangle$$

$$\leq -\sum_{k\in K_*} \breve{\rho}_k(x,x^*)\left\|\theta(x,x^*)\right\|^m. \qquad (3.8)$$

Now, using (3.1), (3.2), and (3.6) - (3.8), we find that

$$\phi(\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] - (\sum_{i=1}^{p} u_i^*[f_i(x^*) - \lambda^* g_i(x^*)]$$

$$-\frac{1}{2}\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \omega(x,x^*)\rangle))$$

$$\geq -[\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*)z^*, \eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*), \pi(x,x^*)\rangle$$

$$+\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) + \frac{1}{2}\sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*)z^*, \eta(x,x^*)\rangle]$$

$$+\frac{1}{2}\langle \sum_{k=1}^{r} w_k^* \nabla H_k(x^*), \pi(x,x^*)\rangle]$$

$$+\sum_{i\in I_+} u_i^*[\bar{\rho}_i(x,x^*) + \lambda^* \tilde{\rho}_i(x,x^*)]\left\|\theta(x,x^*)\right\|^m$$

$$\geq \{\sum_{i\in I_+} u_i^*[\bar{\rho}_i(x,x^*) + \lambda^* \tilde{\rho}_i(x,x^*)] + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x,x^*) + \sum_{k\in K_*} \breve{\rho}_k(x,x^*)\}\left\|\theta(x,x^*)\right\|^m$$

$(by$ (3.7) and (3.8))

≥ 0 $(by\ (iv))$.

But $\phi(a) \geq 0 \Rightarrow a \geq 0$, we have

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] - (\sum_{i=1}^{p} u_i^*[f_i(x^*) - \lambda^* g_i(x^*)]$$

$$-\frac{1}{2}\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)], \omega(x,x^*)\rangle) \geq 0, \qquad (3.9)$$

which using (3.3) implies that

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq 0.$$

Now using this inequality and Lemma 3.1, we have

$$\varphi(x^*) = \lambda^* \leq \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u\in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} = \varphi(x).$$

Since $x \in \mathbb{F}$ is arbitrary, we conclude from this inequality that x^* is an optimal solution to (P).

(b): Proceeding as in part (a), for each $j \in J_+$, we have

$$(v_j^* G_j(x) \leq 0 \leq v_j^* G_j(x^*) - \frac{1}{2}\langle \omega(x,x^*), v_j^* \nabla G_j(x^*)\rangle,$$

which implies

$$\hat{\phi}_j(v_j^* G_j(x) - v_j^* G_j(x^*) + \frac{1}{2}\langle \omega(x,x^*), v_j^* \nabla G_j(x^*)\rangle) \leq 0,$$

which in view of (ii) implies that

$$\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \frac{1}{2}\sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*)z^*, \eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*), \pi(x,x^*)\rangle \leq -\hat{\rho}(x,x^*)\left\|\theta(x,x^*)\right\|^m.$$

Now proceeding as in the proof of part (a) and using this inequality instead of (3.6), we arrive at (3.8), which leads to the desired conclusion that x^* is an optimal solution of (P).

(c) - (e): The proofs are similar to those of parts (a) and (b).

(f): Since $\rho(x,x^*)\geq 0$, (3.1) and (3.2) imply

$$\langle\eta(x,x^*),\nabla L(x^*,u^*,v^*,w^*,\lambda^*)+\frac{1}{2}\nabla^2 L(x^*,u^*,v^*,w^*,\lambda^*)z^*\rangle$$

$$+\langle\pi(x,x^*),\nabla L(x^*,u^*,v^*,w^*,\lambda^*)\rangle\geq-\rho(x,x^*)\left\|\theta(x,x^*)\right\|^m,$$

which in view of our (ϕ, η, ω, π, ρ, θ, m)-pseudosonvexity assumption implies that

$$\phi(L(x,u^*,v^*,w^*,\lambda^*)-[L(x^*,u^*,v^*,w^*,\lambda^*)$$
$$-\frac{1}{2}\langle\nabla L(x^*,u^*,v^*,w^*,\lambda^*),\omega(x,x^*)\rangle])\geq 0.$$

But $\phi(a)\geq 0\Rightarrow a\geq 0$ and hence we have

$$L(x,u^*,v^*,w^*,\lambda^*)\geq 0.$$

Because $x,x^*\in\mathbb{F},v^*\geq 0$, and (3.3), (3.3) and (3.5) hold, we get

$$\sum_{i=1}^{p}u_i^*[f_i(x)-\lambda^* g_i(x)]\geq 0.$$

As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* is an optimal solution of (P).

Theorem 3.2. Let $x^*\in\mathbb{F}$, $\lambda^*=\varphi(x^*)$, the functions $f_i,g_i,i\in \underline{p},G_j,j\in\underline{q}$, and $H_k,k\in\underline{r}$, be twice differentiable at $x^{*.}$ Then there exist $u^*\in U,v^*\in\mathbb{R}_+^q$, and $w^*\in\mathbb{R}^r$ such that (3.1) - (3.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-pseudosonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) for each $j\in J_+\equiv J(v^*),G_j$ is ($\hat{\phi}_j$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*,\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;

(iii) for each $k\in K_*\equiv K(w^*),\xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^*, and $\breve{\phi}_k(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)\geq 0$ for all $x\in\mathbb{F}$;

(b) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-pseudosonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to C(\xi,v^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*,\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) for each $k\in K_*,\xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^*, and $\breve{\phi}_k(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)\geq 0$ for all $x\in\mathbb{F}$;

(c) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-pseudosonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) for each $j\in J_+,G_j$ is ($\hat{\phi}_m$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*,\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;

(iii) $\xi\to D(\xi,w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonvex at x^*, and $\breve{\phi}(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\breve{\rho}(x,x^*)\geq 0$ for all $x\in\mathbb{F}$;

(d) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-pseudosonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to C(\xi,v^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*,\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) $\xi\to D(\xi,w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonv

(e) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-pseudosonvex at x^*, and

(ii) $\xi\to g(\xi,v^*,w^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*,\hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)\geq 0$ for all $x\in\mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a): Based on assumptions specified in (ii) and (iii), (3.6) - (3.8) still hold for this case. From (3.1), (3.2), (3.6), (3.7), and (iv) we deduce that

$$\langle\sum_{i=1}^{p}u_i^*[\nabla f_i(x^*)-\lambda^*\nabla g_i(x^*)]+\frac{1}{2}\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)$$

$$-\lambda^*\nabla^2 g_i(x^*)]z^*,\eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle\sum_{i=1}^{p}u_i^*[\nabla f_i(x^*)-\lambda^*\nabla g_i(x^*)],\pi(x,x^*)\rangle$$

$$\geq -[\langle\sum_{j=1}^{q}v_j^*\nabla G_j(x^*)+\frac{1}{2}\sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)z^*,\eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle\sum_{j=1}^{q}v_j^*\nabla G_j(x^*),\pi(x,x^*)\rangle$$

$$+\langle\sum_{k=1}^{r}w_k^*\nabla H_k(x^*)+\frac{1}{2}\sum_{k=1}^{r}w_k^*\nabla^2 H_k(x^*)z^*,\eta(x,x^*)\rangle$$

$$+\frac{1}{2}\langle\sum_{k=1}^{r}w_k^*\nabla H_k(x^*),\pi(x,x^*)\rangle]$$

$$\geq [\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)]\left\|\theta(x,x^*)\right\|^m \quad (by\ (3.6)\ and\ (3.7))$$

$$\geq -\bar{\rho}(x,x^*)\left\|\theta(x,x^*)\right\|^m \quad (by\ (iv)),$$

which in view of (i) implies that

$$\bar{\phi}(\varepsilon(x,u^*,\lambda^*)-[\varepsilon(x^*,u^*,\lambda^*)-\frac{1}{2}\langle\nabla\varepsilon(x^*,u^*,\lambda^*),\omega(x,x^*)\rangle])\geq 0.$$

Based on the properties of the function $\bar{\phi}$, the last inequality yields

$$\varepsilon(x,u^*,\lambda^*)\geqq 0.$$

As shown in the proof of Theorem 3.1, this inequality leads to the conclusion that x^* is an optimal solution to (P).

(b) - (e) : The proofs are similar to that of part (a).

Theorem 3.3. Let $x^*\in\mathbb{F}$, let $\lambda^*=\varphi(x^*)$, and assume that the functions $f_i, g_i, i\in \underline{p}, G_j, j\in\underline{q}$, and $H_k, k\in\underline{r}$, are twice differentiable at x^*, and that there exist $u^*\in U, v^*\in\mathbb{R}_+^q$, and $w^*\in\mathbb{R}^r$ such that (3.1) - (3.4) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is prestrictly ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-quasisonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) for each $j\in J_+\equiv J_+(v^*), G_j$ is ($\hat{\phi}_j$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*, \hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;

(iii) for each $k\in K_*\equiv K(w^*), \xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^*, and $\breve{\phi}_k(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)>0$ for all $x\in\mathbb{F}$;

(b) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is prestrictly ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-quasisonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to C(\xi,v^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*, \hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) for each $k\in K_*, \xi\to D_k(\xi,w^*)$ is ($\breve{\phi}_k$, η, ω, π, $\breve{\rho}_k$, θ, m)-quasisonvex at x^*, and $\breve{\phi}_k(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)>0$ for all $x\in\mathbb{F}$;

(c) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is prestrictly ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-quasisonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) for each $j\in J_+, G_j$ is ($\hat{\phi}_j$, η, ω, π, $\hat{\rho}_j$, θ, m)-quasisonvex at $x^*, \hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0)=0$;

(iii) $\xi\to D(\xi,w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonvex at x^*, and $\breve{\phi}(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\breve{\rho}(x,x^*)>0$ for all $x\in\mathbb{F}$;

(d) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is prestrictly ($\bar{\phi}$, η^*, ω, π, $\bar{\rho}$, θ, m)-quasisonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to C(\xi,w^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ,m)-quasisonvex at $x^*, \hat{\phi}$ is increasing, and $\hat{\phi}(0)=0$;

(iii) $\xi\to D(\xi,w^*)$ is ($\breve{\phi}$, η, ω, π, $\breve{\rho}$, θ, m)-quasisonvex at x^*, and $\breve{\phi}(0)=0$;

(iv) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)+\breve{\rho}(x,x^*)>0$ for all $x\in\mathbb{F}$;

(e) (i) $\xi\to\varepsilon(\xi,u^*,\lambda^*)$ is prestrictly ($\bar{\phi}$, η, ω, π, $\bar{\rho}$, θ, m)-quasisonvex at x^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi \to g(\xi, v^*, w^*)$ is ($\hat{\phi}$, η, ω, π, $\hat{\rho}$, θ, m)-quasisonvex at $x^*, \hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. The proof is similar to that of Theorem 3.2.

4. CONCLUDING REMARKS

We established several results applying the new notion of higher order (ϕ, η, ω, π, ρ, θ, m)-invexities, which generalizes/unifies most of the existing generalized invexities and its variants in the literature, and then we proved some results on second-order optimality conditions for our principal problem (P). The obtained results to the context of discrete minmax fractional programming offer further applications to other fields of research endeavors relating to discrete fractional programming problems, including the publications [1-4] relating to the multitime multiobjective variational problems.

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