

The Effects of Classical Trapping on the Control of Malaria Transmission

Zhenbu Zhang, Tor A. Kwembe and Xing Yang*

Department of Mathematics and Statistical Sciences, Jackson State University, Jackson, MS 39217, USA

Abstract: This paper investigates the effects of classical trapping on the control of malaria transmission. The Ross-Macdonald model is modified and a trapping probability function is introduced to construct a partial differential equation (PDE) system. The proof of existence and uniqueness of solution of density functions to the PDE system is given, numerical simulation results based on Gaussian distribution and exponential distribution are obtained for the solutions, and graphical representations of solutions are shown and interpreted.

Keywords: Trapping Probability, Ross-Macdonald Model, Gaussian Distribution, Mosquito Density, Malaria.

1. INTRODUCTION

Malaria is one of the most devastating disease and a leading cause of death in the tropical regions of the world [1]. Half the world's population is at risk for malaria, which is endemic in more than 100 countries. Although preventable and treatable, malaria causes significant morbidity and mortality, particular in resource-poor regions [2].

Malaria is an infectious disease caused by the *Plasmodium* parasite and transmitted between humans through the bite of female *Anopheles* mosquito [3]. The incidence of malaria has been growing recently due to increasing parasite drug-resistance and mosquito insecticide-resistance.

Malaria is spread in three ways. The most common way is by the bite of an infected female *Anopheles* mosquito. Although malaria could also be spread through a transfusion of infected blood and by sharing needles with an infected person, they can be prevented effectively. Therefore, as long as we can find an effective preventive measure to prevent the first way that malaria can spread, malaria can be reduced or eradicated. Although, in some tropical regions, malaria has decreased recently, in some areas, the transmission of the disease is still a severe threat and the factors that maintain the transmission remain little understood. Therefore, it is very important to investigate these factors thoroughly by developing and studying appropriate mathematical models to establish the essential tools and identifiable targets needed to eliminate the transmission of malaria.

Various control strategies have been applied to reduce malaria transmission.

Different mathematical models have been developed to investigate the effects of these strategies. In [4], the authors investigated the effects of indoor residual spraying. In [5], the authors investigated the effects of migration on malaria transmission. In [6], the authors investigated the effects of vaccinations. In [7], the authors investigated the effects of weather on malaria transmission. In [8], the authors investigated the impact of chemo-therapy on optimal control of malaria transmission. In [9], the authors investigated the effects of anti-malaria drugs. In this paper, we will investigate the effects of classical trapping on the control of malaria transmission. It is well known that classical trapping problems are related in many ways to search problems when the target moves as a diffusion process [10]. These results will provide the decision maker some useful references to take appropriate control or preventive measures.

2. DESCRIPTION OF THE MODEL

We will consider a simple modification of Ross-Macdonald model. Because the life expectancy of a human is much longer than that of a mosquito we assume that the population of humans is closed with no births and no deaths except from malaria. We also assume that humans and mosquitoes are either infected or uninfected and the total numbers of humans and mosquitoes are constants. Thus we need only to investigate the dynamics of the infected humans and mosquitoes.

Let $u(t, x)$ and $v(t, x)$ be the spatial densities of infected humans and infected mosquitoes at time t in x , respectively. Let a be the human-biting rate, that is, the rate at which mosquitoes bite humans and b be

*Address correspondence to this author at the Department of Mathematics and Statistical Sciences, Jackson State University, Jackson, MS 39217, USA; Fax: 601.979.5852; E-mail: xing.yang@jsums.edu

the mosquito-to-human transmission efficiency, that is, the probability, given an infectious mosquito has bitten a susceptible human, that the human becomes infected. ρ denotes the human-to-mosquito transmission efficiency, that is, the probability, given a susceptible mosquito has bitten an infectious human, that the mosquito becomes infected. $r = \frac{M}{N}$ denotes the ratio of the number of female mosquitoes to humans, M is the size of female mosquito population and N is the size of human population. We assume that M and N are constants. Malaria is strongly associated with location, with disease transmission restricted to a few kilometers from specific mosquito breeding sites [4, 11]. Same as in [9, 10], we assume that the flight of the mosquitoes and the movement of humans are supposed to be a Brownian motion so that a classical diffusion term appears in the equations governing the mosquito and human density. Assume the diffusion rates for humans and mosquitoes are d_1 and d_2 , respectively. μ denotes the recovery rate of human hosts due to treatment and δ denotes the per capita death rate of infected human hosts due to the disease. η is the mosquito death rate. Then one version of mathematical model for malaria transmission in a region Ω with diffusion is

$$\begin{cases} \frac{du}{dt} = d_1 \Delta u + abrv(1-u) - (\mu + \delta)u, & x \in \Omega, t > 0, \\ \frac{dv}{dt} = d_2 \Delta v + \rho u(1-v) - \eta v, & x \in \Omega, t > 0. \end{cases} \quad (2.1)$$

In this paper, we are going to investigate the effects of classical trapping on malaria transmission. Associated with the trap is a "trap radius" r_i with the property that if distance between the mosquito and the center of the trap is less than r_i , then the mosquito is trapped [12]. Same as in [10] and [12], we define the probability density for an untrapped mosquito as

$$f(x, y, t) dx dy = Pr \left\{ \begin{array}{l} \text{at time } t \text{ the mosquito is in a small area } dx dy \\ \text{around the point } (x, y) \text{ and is not trapped} \end{array} \right\}.$$

Therefore, the probability that a mosquito is not trapped at time t is

$$P(t) = \int_{\Omega} f(x, y, t) dx dy,$$

and the probability that is trapped is

$$1 - P(t) = 1 - \int_{\Omega} f(x, y, t) dx dy.$$

Also same as in [10] and [12], we define the trapping function $\Phi(x, y, t, z)$ as

$$\Phi(x, y, t, z) dt = Pr \left\{ \begin{array}{l} \text{mosquito is trapped by one of the } N \text{ traps} \\ \text{in the time interval } (t, t + \Delta t) \text{ at time } t \text{ the} \\ \text{mosquito is located at } (x, y) \text{ and the vector of trap} \\ \text{location is } z \end{array} \right\},$$

where

$$z = (z_{x_1}, z_{y_1}, z_{x_2}, z_{y_2}, \dots, z_{x_N}, z_{y_N}),$$

and (z_{x_i}, z_{y_i}) are the coordinates of the center of the i -th trap. We adopt one model for the trapping function as follows

$$\Phi(x, y, z) = \sum_{i=1}^N a_i [q_i + (x - z_{x_i})^2 + (y - z_{y_i})^2]^{-\sigma},$$

where a_i , q_i , and σ are positive parameters. Assume that the distribution of displacement in a short time Δt is Gaussian with mean displacement $0 + o(\Delta t)$ and variance $D\Delta t + o(\Delta t)$, then we have

$$\frac{\partial f}{\partial t} = D\Delta f - \Phi f. \quad (2.2)$$

Thus we have the following model:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + abrv(1-u) - (\mu + \delta)u, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \rho u(1-v) - \eta v - (1 - \int_{\Omega} f(x, y, t) dx dy)v, \end{cases} \quad (2.3)$$

where $f(x, y, t)$ satisfies (2.2).

Let $\alpha = abrv$, $\beta = \rho$, and $g(t) = 1 - \int_{\Omega} f(x, y, t) dx dy \geq 0$, then we can rewrite (2.3) as

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + \alpha v(1-u) - (\mu + \delta)u, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \beta u(1-v) - \eta v - g(t)v, \end{cases} \quad (2.4)$$

In this paper we consider an experimental region $\Omega = B(0, R) \subset R^2$ which is a disc centered at origin with radius R (Ω could be the whole R^2 plane). If $\Omega = R^2$, we assume that $u(x, y), v(x, y) \rightarrow 0$ as $(x, y) \rightarrow \infty$. When Ω is bounded, we assume that mosquitoes and humans do not enter or leave the region. This means u and v satisfy the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \text{ on } \partial\Omega,$$

where ν is the outward normal to the boundary $\partial\Omega$ of the region Ω .

3. EXISTENCE OF PROBABILITY DENSITY FUNCTION

3.1. $\Omega = R^2$

We first consider a special case. That is, $\Omega = R^2$ and there is a single trap located at the origin. For the convenience of notations, we express all functions in terms of polar coordinates. The trap function is

$$\Phi(r) = \frac{a}{(q+r^2)^\sigma},$$

and the associated problem for f becomes an initial value problem

$$\begin{cases} \frac{\partial f}{\partial t} - D\Delta f = -\Phi f, & r_i < r < +\infty, t > 0, \\ f(r, 0) = f_0, \end{cases} \quad (3.1)$$

where

$$f_0 = \begin{cases} \frac{1}{\pi(r^2 - r_i^2)}, & r > r_i, \\ 0, & \text{otherwise.} \end{cases}$$

For initial value problem (3.1), the authors in [10] proved the local existence and uniqueness under the assumptions $\Phi, f_0 \in L^{2/\theta}(R^2)$ with $0 < \theta < 1$ and weak global existence and uniqueness under the assumptions that $\Phi = \phi^2$ such that

$$\phi \in L^{4/\theta, 4/(2-\theta)}(R^2 \times R_+), \quad 2/3 < \theta < 4/3,$$

and

$$g = W * f_0 \in L^{4/\theta, 4/(2-\theta)}(R^2 \times R_+),$$

where W is the fundamental solution of heat equation and $*$ denotes the convolution in space.

3.2. $R < \infty$

When $R < \infty$, that is, Ω is bounded, the related problem for f is an initial-boundary value problem. Again, we assume that there is a single trap located at the origin. Then the problem is

$$\begin{cases} \frac{\partial f}{\partial t} = D\Delta f - \Phi f, & r_i < r < R, \\ f(r_i, t) = 0, \partial_r f(R, t) = 0, \\ f(r, 0) = f_0. \end{cases} \quad (3.2)$$

It seems there is no existence results available for this problem. Since the boundary condition are

homogeneous, we solve this problem by using separation of variables.

Assume that $f(r, t) = p(r)q(t)$. Recall that assuming radial symmetry, in terms of polar coordinates,

$$\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right).$$

By substituting this into (3.2) we have

$$pq' - Dp''q - \frac{D}{r} p'q = -\Phi(r)pq.$$

Dividing each term by pq and move all terms with q to one side and all terms with p to the other side we have

$$\frac{q'}{q} = \frac{Dp''}{p} + \frac{Dp'}{rp} - \Phi(r).$$

Since the left hand side is a function of t and the right hand side is a function of r and it holds for all t and r , it must be a constant, say, $-\lambda$. Therefore, we have

$$\frac{q'}{q} = \frac{Dp''}{p} + \frac{Dp'}{rp} - \Phi(r) = -\lambda.$$

Solving

$$\frac{q'}{q} = -\lambda$$

gives us

$$q(t) = ce^{-\lambda t}.$$

From the boundary conditions we have

$$p(r_i) = 0, \quad p'(R) = 0.$$

Thus $p(r)$ satisfies

$$\begin{cases} Dp'' + Dp' - \Phi(r)rp + \lambda rp = 0, & r_i < r < R, \\ p(r_i) = 0, \quad p'(R) = 0. \end{cases} \quad (3.3)$$

This is a standard regular Sturm-Liouville eigenvalue problem [13]. It is well known that there exist an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Corresponding each eigenvalue λ_n , there is an eigenfunction $\phi_n(r)$ and $\{\phi_n(r), n = 1, 2, \dots\}$ form a complete set. Thus we assume that

$$f(r, t) = \sum_{n=1}^{\infty} A_n \phi_n(r) e^{-\lambda_n t}.$$

From the initial condition, we have

$$\sum_{n=1}^{\infty} A_n \phi_n(r) = f_0.$$

Therefore,

$$A_n = \frac{\int_{r_1}^R \phi_n(r) f_0(r) r dr}{\int_{r_1}^R \phi_n^2(r) r dr}.$$

4. EXISTENCE AND UNIQUENESS OF SOLUTION

Now we prove the existence and uniqueness of solution of (2.4) for $R < \infty$ by constructing a pair of super- and sub-solutions. For a bounded region Ω , (2.4) can be formularized as

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + \alpha v(1-u) - (\mu + \delta)u, t > 0, x \in \Omega, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \beta u(1-v) - \eta v - g(t)v, t > 0, x \in \Omega, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} = 0, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), x \in \Omega. \end{cases} \quad (4.1)$$

For the convenience of notations we let

$$f_1(t, u, v) = \alpha v(1-u) - (\mu + \delta)u,$$

$$f_2(t, u, v) = \beta u(1-v) - \eta v - g(t)v.$$

It is easily seen that

$$f_{1v} = \alpha(1-u) \geq 0, f_{2u} = \beta(1-v) \geq 0.$$

Therefore, system (4.1) is a quasi-monotonic increasing system. For such a system, a pair of functions $U = (\bar{u}, \bar{v})$, $V = (\underline{u}, \underline{v})$ is called a pair of super- and sub- solutions of (4.1) if

$$\frac{\partial \bar{u}}{\partial t} - d_1 \Delta \bar{u} - f_1(\bar{u}, \bar{v}) \geq 0 \geq \frac{\partial \underline{u}}{\partial t} - d_1 \Delta \underline{u} - f_1(\underline{u}, \underline{v}),$$

$$\frac{\partial \bar{v}}{\partial t} - d_2 \Delta \bar{v} - f_2(\bar{u}, \bar{v}) \geq 0 \geq \frac{\partial \underline{v}}{\partial t} - d_2 \Delta \underline{v} - f_2(\underline{u}, \underline{v}),$$

$$\frac{\partial \bar{u}}{\partial \nu} \Big|_{\partial \Omega} \geq 0 \geq \frac{\partial \underline{u}}{\partial \nu} \Big|_{\partial \Omega}, \frac{\partial \bar{v}}{\partial \nu} \Big|_{\partial \Omega} \geq 0 \geq \frac{\partial \underline{v}}{\partial \nu} \Big|_{\partial \Omega},$$

$$\bar{u}(x, 0) \geq u_0(x) \geq \underline{u}(x, 0), \bar{v}(x, 0) \geq v_0(x) \geq \underline{v}(x, 0).$$

It is well-known that if (4.1) has a pair of super- and sub- solutions $U(x, t)$ and $V(x, t)$ such that $V(x, t) \leq U(x, t)$, then it has a unique solution in $[V(x, t), U(x, t)]$ [14]. Now we prove that (4.1) has a solution by constructing a pair of super- and sub-solutions $U(x, t)$ and $V(x, t)$ as follows.

First, it is easily seen that $U = (\bar{u}, \bar{v}) = (1, 1)$ and $V = (\underline{u}, \underline{v}) = (0, 0)$ is a pair of super- and sub- solutions. Therefore, (4.1) has a solution. In order to get a better idea about the solution, we try to look for a nonconstant super-solution.

Assume $(1, 1) \geq U = (\bar{u}, \bar{v}) = (p(t), q(t)) > (0, 0)$ is a super-solution. Since it is independent of x , the boundary conditions are automatically satisfied. To become a super-solution, $p(t)$ and $q(t)$ should satisfy

$$\begin{cases} p' + (\mu + \delta)p \geq \alpha q(1-p), \\ q' + (\eta + g(t))q \geq \beta p(1-q). \end{cases} \quad (4.2)$$

Since $p \leq 1$ and $q \leq 1$, to have (4.2), we need only

$$\begin{cases} p' + (\mu + \delta + \alpha)p = \alpha, \\ q' + (\eta + \beta + g(t))q = \beta. \end{cases} \quad (4.3)$$

Solving (4.3) gives

$$p(t) = \left(u_0 - \frac{\alpha}{\mu + \delta + \alpha}\right) e^{-(\mu + \delta + \alpha)t} + \frac{\alpha}{\mu + \delta + \alpha},$$

$$q(t) = e^{-(\eta + \beta)t - \int_0^t g(\tau) d\tau} \left(\beta \int_0^t e^{(\eta + \beta)\tau + \int_0^\tau g(s) ds} d\tau + v_0 \right),$$

where $u_0 = \max_{\bar{\Omega}} u_0(x)$, $v_0 = \max_{\bar{\Omega}} v_0(x)$. Therefore, we know that (4.1) has a unique solution $(u(x, t), v(x, t))$ satisfying

$$0 \leq u(x, t) \leq p(t), \quad 0 \leq v(x, t) \leq q(t).$$

From the expression of $p(t)$ and $q(t)$ we can get some long time estimate for $(u(x, t), v(x, t))$ which can also been seen from the numerical simulations we will do next.

5. NUMERICAL SIMULATION RESULTS

In this section we adopt realistic probability distributions to simulate the initial conditions of both u and v , assign values to parameters, solve the PDE system (4.1) numerically and graph the solutions of u and v .

Assume both u and v follow a Gaussian distribution at time $t = 0$, we can write the initial conditions as follows:

$$u(x,0) = 1 / (\sigma\sqrt{2\pi}) \exp[-x^2 / (2\sigma^2)], x \in \Omega,$$

$$v(x,0) = 1 / (\sigma\sqrt{2\pi}) \exp[-x^2 / (2\sigma^2)], x \in \Omega.$$

Also we may consider to take $g(t) = 1 - e^{-t}$ since the derivative of $g(t)$ actually should be an exponential distribution which implies that half of mosquitos will have been trapped at time $t=1$ and all mosquitos will be trapped at $t = \infty$.

The parameters values we use in the numerical simulation are:

$$d_1 = 1, d_2 = 0.01, \alpha = 0.2, \beta = 0.1, \mu = 0.1, \delta = 0.05, \eta = 0.05.$$

For simplicity we also set the radius of Ω to 1, then we can set σ to $1/6$ because the probability falling within three standard deviations of a Gaussian distribution is almost 100%. This actually assumes almost 100% of the initial infected humans $u(x,0)$ and infected mosquitos $v(x,0)$ are within the region of radius of 1 at $t = 0$.

Under polar coordinates system we can use the NDSolve function in Mathematica to obtain numerical solutions of $u(x,t)$ and $v(x,t)$ and plot their graphs as follows.

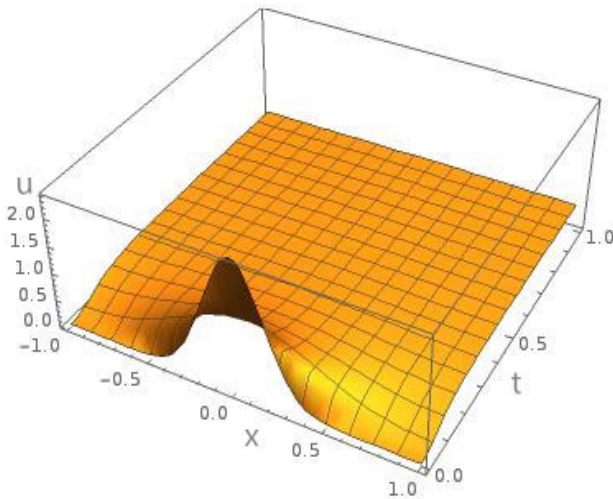


Figure 1: Density of Infected Humans with Time t and Distance x from Center.

From Figure 1 we see that the density of infected humans has largest value at the center which is a natural consequence since the initial condition (Gaussian distribution) already assumes the maximum density occurs at the center at the beginning. However, the density of infected humans drops very quickly with time t. We observe that it drops to close to 0 at time $t = 1$.

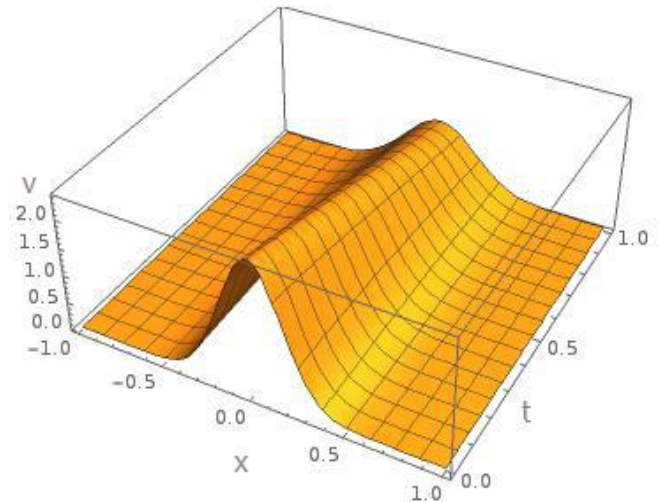


Figure 2: Density of Infected Mosquitos with Time t, Distance x from Center and Trap Probability $g(t) = 1 - e^{-t}$.

From Figure 2 we see that the density of infected mosquitos also has largest value at the center which is a natural consequence since the initial condition (Gaussian distribution) also assumes the maximum density occurs at the center at the beginning. We notice that the density of infected mosquitos dropped not so quickly. It still maintains a significant level at time $t = 1$. This is because we choose the trap probability $g(t) = 1 - e^{-t}$ which has trapped half mosquitos in time $t = 1$. This can be seen clearly if we plot a faster trapping case when choosing trap probability $g(t) = 1 - e^{-10t}$ (Figure 3), and the case without the trap (Figure 4).

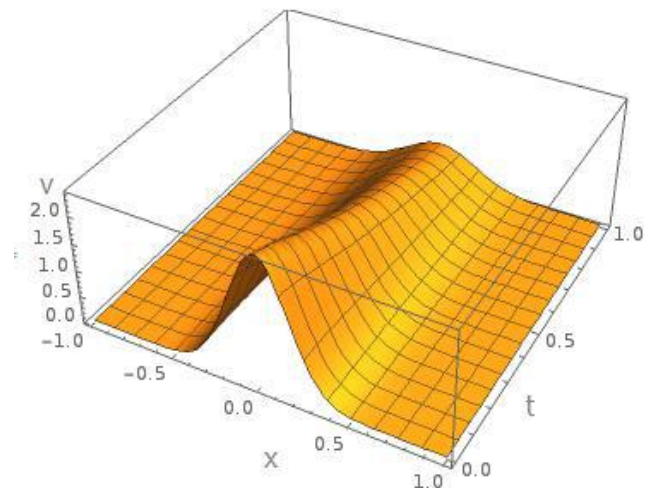


Figure 3: Density of Infected Mosquitos with Time t, Distance x from Center and Trap Probability $g(t) = 1 - e^{-10t}$.

From Figure 3 we see the density of infected mosquitos drop a lot at a very fast pace since the trapping coefficient in front of time t now is 10. From

Figure 4 the density drops very slowly without a trap. The three above figures show the big difference how trapping is important to reduce the density of mosquitos.

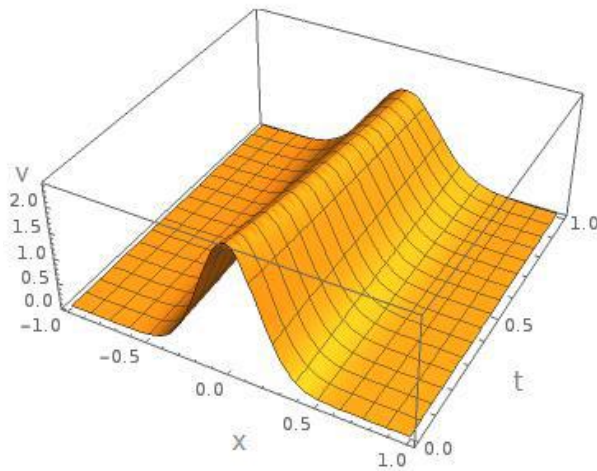


Figure 4: Density of Infected Mosquitos with Time t and Distance x from Center without a Trap ($g(t)=0$).

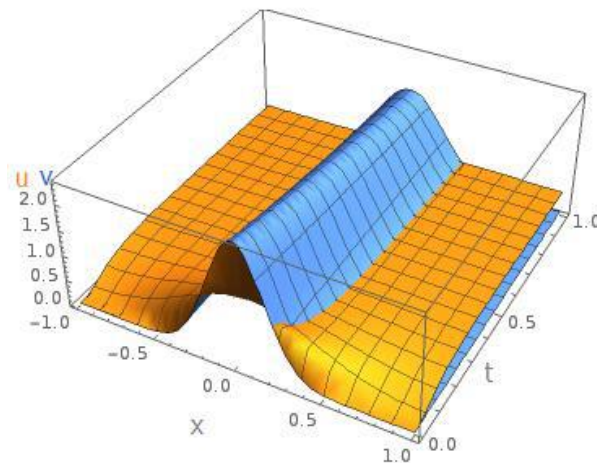


Figure 5: Comparison of Density of Infected Humans with that of Infected Mosquitos.

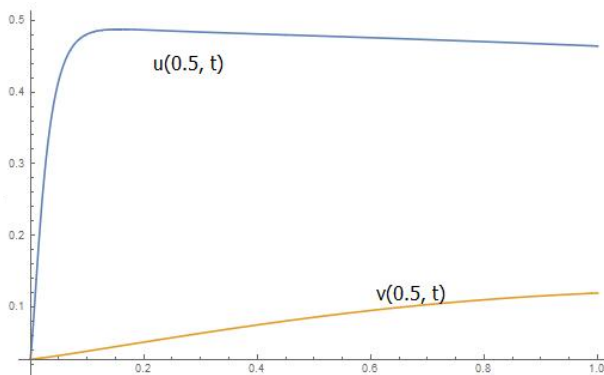


Figure 6: Comparison of Density of Infected Humans with that of Infected Mosquitos for $x = 0.5$ (Note that the plot of $x = -0.5$ is same due to symmetry).

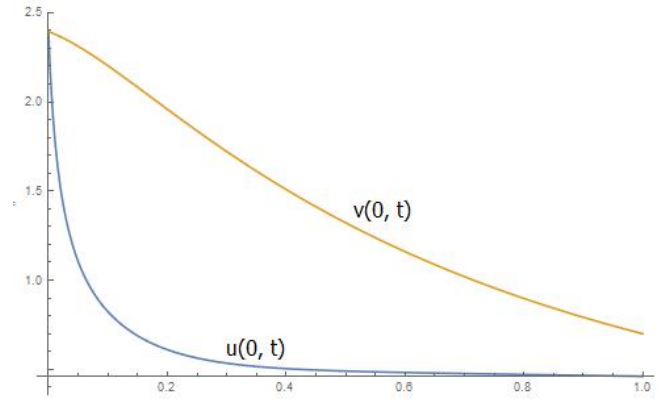


Figure 7: Comparison of Density of Infected Humans with that of Infected Mosquitos for $x = 0$.

We can combine two density plots Figures 1 and 4 into Figure 5 to make a comparison. The density of infected humans is in yellow and density of infected mosquitos is in blue. Apparently the density of infected mosquitos around the center ($x = 0$) is higher than that of infected humans and continue to maintain high level with time t . To see this clearly, two dimensionla plots Figures 6 and 7 are provided to compare $u(x,t)$ with $v(x,t)$ for $x = \pm 0.5$ and $x = 0$ respectively. (Please note that due to symmetry plot of $x = -0.5$ is same as that of $x = 0.5$.) Therefore, to eliminate malaria infection of a fixed region quickly, these simulation results suggest that people may not spend all time and medical resources all over the region, instead they should focus on center places where infected mosquitos are heavily populated. More importantly, a powerful trap with long trap radius is the key factor to reduce density of mosquitos thus control malaria transmission timely and efficiently.

ACKNOWLEDGEMENT

This research is supported by National Science Foundation under Grant No. DMS1330801.

REFERENCES

- [1] King AT, Mends-Brew E, Osei-Frimpong E, Ohene KR. Mathematical model for the control of malaria-Case study: Chorkor polyclinic, Accra, Ghana, Global. Adv Res J Med Med Sci 2012; 1: 108-18.
- [2] U.S. Global Health Policy, Fact Sheet, March 2011, The Henry J. Kaiser Family Foundation. [cited 2016 May 1]: Available from: http://kff.org/topic/global-health-policy/?post_type=fact-sheet
- [3] Chitnis N, Cushing JP, Hyman JM. Bifurcation analysis of a mathematical model for malaria transmission. SIAM J Appl Math 2006; 67: 24-45. <https://doi.org/10.1137/050638941>
- [4] Al-Arydah M, Smith R. Controlling malaria with indoor residual spraying in spatially heterogeneous environments. Math Biosci Eng 2011; 8: 889-914. <https://doi.org/10.3934/mbe.2011.8.889>

- [5] Prosper O, Ruktanonchai N, Martcheva M. Assessing the role of spatial heterogeneity and human movement in malaria dynamics and control. *J Theor Biol* 2012; 303: 1-14. <https://doi.org/10.1016/j.jtbi.2012.02.010>
- [6] Koella JC. On the use of mathematical models of malaria transmission. *Acta Tropica* 1991; 49: 1-25. [https://doi.org/10.1016/0001-706X\(91\)90026-G](https://doi.org/10.1016/0001-706X(91)90026-G)
- [7] Hoshen MC, Morse AP. A weather-driven model of malaria transmission. *Malar J* 2004; 3: 32. <https://doi.org/10.1186/1475-2875-3-32>
- [8] Makinde OD, Okosun KO. Impact of chemo-therapy on optimal control of malaria disease with infected immigrants. *BioSyst* 2011; 104: 32-41. <https://doi.org/10.1016/j.biosystems.2010.12.010>
- [9] Bacaër N, Sokhna C. A reaction-diffusion system modeling the spread of resistance to an antimalarial drug. *Math Bios Eng* 2005; 2: 227-38. <https://doi.org/10.3934/mbe.2005.2.227>
- [10] Calderón CP, Kwembe TR. On the classical trapping problem. *Math Bios* 1990; 102: 183-90. [https://doi.org/10.1016/0025-5564\(90\)90061-3](https://doi.org/10.1016/0025-5564(90)90061-3)
- [11] Carter R, Mendis KN, Roberts D. Spatial targeting of interventions against malaria. *Bull WHO* 2000; 78: 1401-11.
- [12] Mangel M. Information and area wide control in agriculture ecology: the classical trapping problem and its extensions. in Levin AS, Hallam TG, Gross LJ, editors. *Biomathematics texts*. New York: Springer-Verlag 1989; pp. 81-112.
- [13] Haberman R. *Applied differential equations with Fourier series and boundary value problems*. 5th ed. New Jersey: Pearson Education 2012.
- [14] Ye QX, Li ZY. *Introduction to reaction-diffusion equations*. Beijing: Scientific Publisher of China 1994.

Received on 31-10-2016

Accepted on 07-11-2016

Published on 18-11-2016

<http://dx.doi.org/10.6000/1927-5129.2016.12.67>

© 2016 Zhang *et al.*; Licensee Lifescience Global.

This is an open access article licensed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted, non-commercial use, distribution and reproduction in any medium, provided the work is properly cited.