Edge Version of Harmonic Index and Harmonic Polynomial of Rooted Product Graphs

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Abstract: In this paper we computed explicit formulas for the edge version of harmonic index and harmonic polynomial of some classes of rooted product of graphs and *i*-th vertex rooted product of graphs.

Keywords: Topological indices, line graph, harmonic index and harmonic Polynomial.

1. INTRODUCTION AND PRELIMINARIES

A graph G consists of two types of elements, namely vertices and edges. A vertex is simply drawn as a node or a dot. The vertex set of *G* is usually denoted by *V*(*G*), or *V* when there is no danger of confusion. The order of a graph is the number of its vertices, i.e. |*V*(*G*)|. An edge (a set of two elements) is drawn as a line connecting two vertices, called end points or end vertices. An edge with end vertices x and y is denoted by xy (without any symbol in between). The edge set of *G* is usually denoted by *E*(*G*), or *E* when there is no danger of confusion, edge *xy* is called incident to a vertex when this vertex is one of the endpoints *x* or *y*. The size of a graph is the number of its edges, i.e. |*E*(*G*)|. For more details see [1]. Let *G* = (*V*; *E*) be a connected simple graph with |*V*| vertices and |*E*| edges. The degree d_v of a vertex *v* is the number of vertices joining to *v*. A line graph *L*(*G*) of a simple graph *G* is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if the corresponding edges of *G* have a vertex in common ([2]). In chemical graph theory line graphs are very useful. In 1981, Bertz introduced the first topological index on the basis of line graph see [3]. For more details on the importance of line graph in chemistry see [4-6]. The Randic index is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies. Mathematical properties of this descriptor have also been studied extensively, as summarized in [7]. The Randic index *R*(*G*) is defined as defined by Randic in [8], $R(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{deg(u)deg(v)}}$. He noticed that this index was well correlated with a variety of physico-chemical properties of alkanes: boiling point, enthalpy of formation, surface area and solubility in water, etc. Eventually, this index became one of the most successful molecular descriptors, and scores of its pharmacological and chemical applications have been reported. Mathematical properties of this descriptor have also been studied extensively in [7, 9-11]. In this paper, we consider a closely related variant of the Randic index, named the harmonic index. For a graph *G*, the harmonic index *H*(*G*) is de defined as $H(G)=\sum_{uv\in E(G)}\frac{2}{d_u+d_v}$. This topological index was first appeared in [12], and it can also be viewed as a particular case of the general sum-connectivity index proposed by Zhou and Trinajstic in [13]. For detailed results on the topological indices of graphs the readers may referred to [12, 14-22]. The harmonic polynomial in [23] is defined as follows

$$H(G,x)=\sum_{uv\in E(G)}2x^{d_u+d_v-1}.$$

In ([24]), Nazir *et al*. introduced the notation of edge version of harmonic polynomial and harmonic index as follows:

$$H_e(G,x)=\sum_{ef\in E(L(G))}2x^{d_e+d_f-1}.$$

Similarly the edge version of harmonic index is defined as

$$H_e(G)=\sum_{ef\in E(L(G))}\frac{2}{d_e+d_f}.$$

Clearly $\int_0^1 H_e(G,x)dx=H_e(G).$

The following lemma is helpful for computing the degree of a vertex of line graph.

Lemma 1.1. Let *G* be a graph with $u,v\in V(G)$ and $e=uv\in E(G)$. Then:

$d_e=d_u+d_v-2.$

In order to calculate the number of edges of an arbitrary graph, the following lemma is significant for us.

Lemma 1.2. Let *G* be a graph. Then

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$$\sum_{u \in V(G)} d_u = 2|E(G)|.$$

This is also known as handshaking Lemma.

In this paper we computed edge version of harmonic index and harmonic polynomial for some classes of rooted product graphs and *i*-th rooted product graphs.

2. ROOTED PRODUCT OF GRAPHS

In this section, first we recall the definition of rooted product of graphs. Let H be a labeled graph on n vertices with vertex set $V(H) = \{1,2,\ldots,n\}$ and let G be a sequence of n rooted graphs $G_1, G_2, \ldots, G_n$. Godsil and Mckay in [8] defined the rooted product of H by G, denoted by $H(G) = H(G_1,G_2,\ldots, G_n)$ is a graph obtained by identifying the root vertex of G_i with the i-th vertex of H for all $i = 1, 2,\ldots,n$. In particular, if the components G_i, $i = 1,2;\ldots, n$ are mutually isomorphic to K, the rooted product of H by G is denoted by $H\{K\}$ and called the cluster product of H by K.

Now we will study some examples of rooted product graphs in the context of Harmonic index and polynomial. Let P_n and C_n denotes the path and cycle on n vertices.

2.1. Harmonic Polynomial and Harmonic Index of $C_n\{P_{k+1}\}$

Let $C_n\{P_{k+1}\}$ is the cluster product of C_n by P_{k+1}. The graph of $C_n\{P_{k+1}\}$ is shown in Figure **1**.

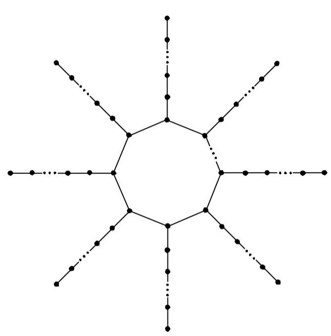

Figure 1: The graph $C_n\{P_{k+1}\}$.

First we consider the graph for $k = 1$. The graph $C_n\{P_2\}$ for $n = 6$, is shown in Figure **2**.

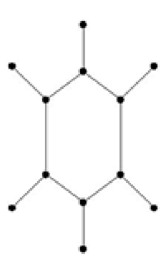

Figure 2: The graph of $C_6\{P_2\}$.

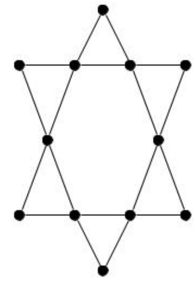

Figure 3: Line graph of $C_6\{P_2\}$.

Proposition 2.1. Let G be the graph of $C_n\{P_2\}$, then

(1) $H_e(G; x) = 2nx^7 + 4nx^5$;

(2) $H(G) = \frac{11}{12}n$.

Proof. In the graph G, the total number of vertices and edges are $2n$ and $2n$ respectively (see Figure **2**). Therefore in line graph $L(G)$, the total number of vertices are $2n$, out of which n vertices are of degree 2 and n vertices are of degree 4 (see Figure **3**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $3n$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **1**.

Table 1: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(2, 4)	(4, 4)
Number of edges	$2n$	n

Consequently we get $H_e(G; x) = 2nx^7 + 4nx^5$ and $H(G) = \frac{11}{12}n$.

Now we will consider the graph $C_n\{P_{k+1}\}$ for $k = 2$. The graph $C_n\{P_3\}$ for $n = 6$, is shown in Figure **4**.

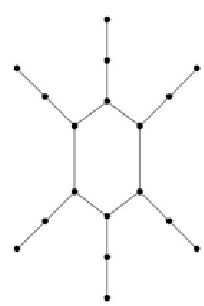

Figure 4: The graph of $C_6\{P_3\}$.

Proposition 2.2. Let G be the graph of $C_n\{P_3\}$, then

(1) $H_e(G; x) = 2nx^7 + 4nx^6 + 2nx^3$;

(2) $H(G) = \frac{37}{28}n$.

Proof. In the graph G, the total number of vertices and edges are $3n$ and $3n$ respectively (see Figure **4**). Therefore in line graph $L(G)$, the total number of vertices are $3n$, out of which n vertices are of degree 1, n vertices are of degree 3 and n vertices are of degree 4 (see Figure **5**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $4n$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **2**.

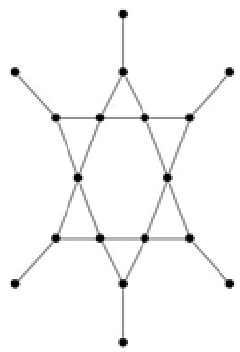

Figure 5: Line graph of $C_6\{P_3\}$.

Table 2: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(1 3)	(3, 4)	(4, 4)
Number of edges	n	$2n$	n

Consequently we get $H_e(G; x)=2nx^7 + 4nx^6 + 2nx^3$ and $H(G) = \frac{37}{28}n$.

Now we will consider the graph $C_n\{P_{k+1}\}$ for $k > 2$. The graph $C_6\{P_4\}$ is shown in Figure **6**.

Proposition 2.3. Let G be a graph $C_n\{P_{k+1}\}$, then

(1) $H_e(G; x) = 2\{nx^7 + 2nx^6 + nx^4 + n(k-3)x^3 + nx^2\}$;

(2) $H_e(G) = \left\{\frac{793}{420} + \frac{1}{2}(k-3)\right\}n$.

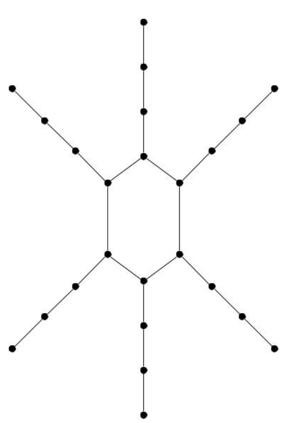

Figure 6: The graph of $C_6\{P_4\}$.

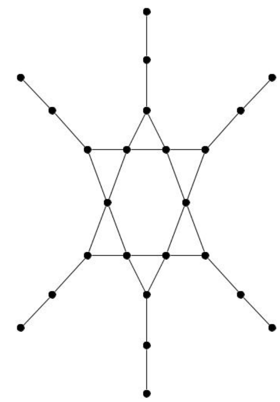

Figure 7: Line graph $C_6\{P_4\}$.

Proof. In the graph G, the total number of vertices and edges are $n(k + 1)$ and $n(k + 1)$ respectively (see Figure **6**). Therefore in line graph $L(G)$, the total number of vertices are $n(k + 1)$, out of which n vertices are of degree 4, n vertices are of degree 1, $n(k\ 2)$ vertices are of degree 2 and n vertices are of degree 3 (see Figure **7**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $n(k + 2)$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **3**.

Table 3: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(1,2)	(2,2)	(2,3)	(3,4)	(4,4)
Number of edges	n	$n(k-3)$	n	$2n$	n

Hence we get $H_e(G; x) = 2\{nx^7 + 2nx^6 + nx^4 + n(k-3)x^3 + nx^2\}$ and $H_e(G) = \left\{\frac{793}{420} + \frac{1}{2}(k-3)\right\}n$.

2.2. Harmonic Polynomial and Harmonic Index of $C_n\{S_2\}$

Let S_n be the star graph having one internal node and n leaves. Consider the cluster product of C_n with S_2 by joining the internal node of the star graph as shown in Figure **8** and is denoted by $C_n\{S_2\}$.

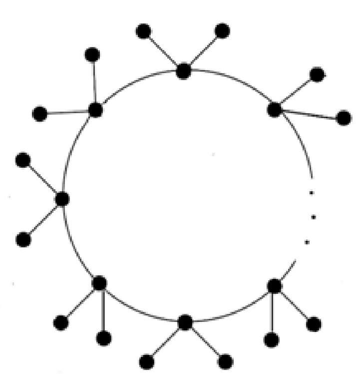

Figure 8: The graph $C_n\{S_2\}$.

Proposition 2.4. Let G be a graph $C_n\{S_2\}$, then

(1) $H_e(G; x) = 2nx^{11} + 8nx^8 + 2nx^5$;

(2) $H_e(G) = \frac{25}{18}n.$

Proof. In the graph G, the total number of vertices and edges are $3n$ and $3n$ respectively (see Figure **9**). Therefore in line graph $L(G)$, the total number of vertices are $3n$, out of

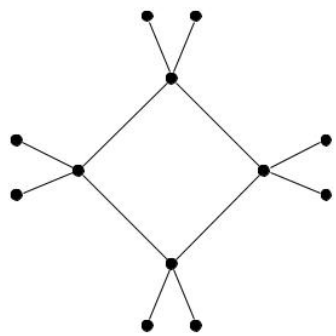

Figure 9: The graph of $C_4\{S_2\}$.

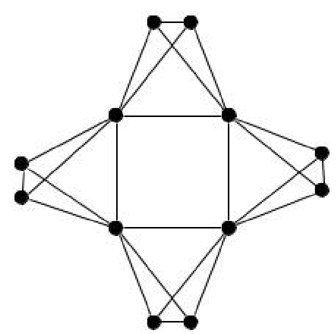

Figure 10: Line graph of $C_4\{S_2\}$.

which n vertices are of degree 6 and $2n$ vertices are of degree 3 (see Figure **10**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $6n$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **4**.

Table 4: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(3,3)	(3,6)	(6,6)
Number of edges	n	$4n$	n

Hence we get $H_e(G, x) = 2nx^{11} + 8nx^8 + 2nx^5$ and $H_e(G) = \frac{25}{18}n.$

3. *i*-th VERTEX ROOTED PRODUCT OF GRAPHS

Motivated by the definition of rooted product of graphs, we define the i-th vertex rooted product of graphs. Let H be a labeled graph on n vertices and let G be a sequence of k rooted graphs $G_1, G_2, \ldots, G_k$. Then the i-th vertex rooted product of H by G, denoted by $H_i\{G_1, G_2, \ldots, G_k\}$ is a graph obtained by identifying the root vertex of every G_l to the i-th vertex of H for all $l = 1, 2, \ldots, k$. In the special case when the components $G_1, G_2, \ldots, G_k$ are mutually isomorphic to a graph L, the i-th vertex rooted product of H by G is denoted by $H^{i;k}\{L\}$ and called the i-th vertex cluster product of H by L. Let $C_n^{i,k}\{P_{u+1}\}$ be the i-th vertex cluster product of C_n by P_{u+1}. It is easy to see that if we replace i-th vertex with some j-th vertex then graph will remain the same, so we denote it by $C_n^k\{P_{u+1}\}$. The graph of $C_n^k\{P_{u+1}\}$ is shown in Figure **11**.

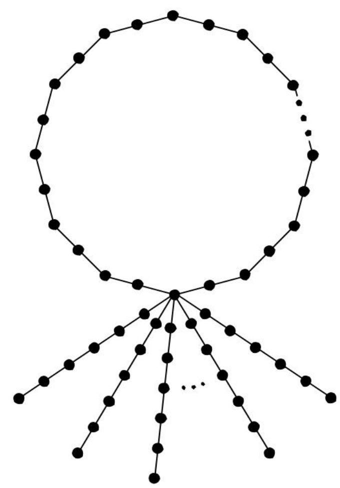

Figure 11: The graph $C_n^k\{P_{u+1}\}$.

3.1. Harmonic Polynomial and Harmonic Index of $C_n^k\{P_{u+1}\}$.

First we will consider the case $C_n^k\{P_{u+1}\}$ for $u = 1$. The graph $C_5^3\{P_2\}$ is shown in Figure **12**.

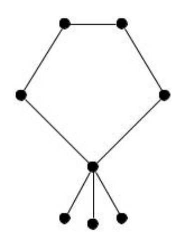

Figure 12: The graph of $C_5^3\{P_2\}$.

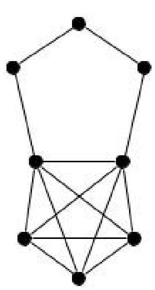

Figure 13: Line graph of $C_5^3\{P_2\}$.

Table 5: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(2, 2)	$(k+1, k+1)$	$(2, k+2)$	$(k+1, k+2)$	$(k+2, k+2)$
Number of edges	n - 3	$\frac{k(k-1)}{2}$	2	$2k$	1

Proposition 3.1. Let G be a graph of $C_n^k\{P_2\}$, then

(1) $H_e(G; x) = 2(n\ 3)x^3 + 2x^{2k+3} + k(k\ 1)x^{2k+1} + 4x^{k+3} + 4kx^{2k+2}$;

(2) $H_e(G) = \frac{1}{2}n - \frac{3}{2} + \frac{2}{2k+4} + \frac{k(k+1)}{2k+2} + \frac{4}{k+4} + \frac{4k}{2k+3}$.

Proof. In the graph G, the total number of vertices and edges are $n + k$ and $n + k$ respectively (see Figure **12**). Therefore in line graph $L(G)$, the total number of vertices are $n + k$, out of which n 2 vertices are of degree 2, 2 vertices are of degree $k + 2$, and k vertices are of degree $k + 1$ (see Figure **13**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $\frac{k^2+3k+2n}{2}$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **5**.

Hence we get $H_e(G; x) = 2(n-3)x^3 + 2x^{2k+3} + k(k-1)x^{2k+1} + 4x^{k+3} + 4kx^{2k+2}$

and $H_e(G) = \frac{1}{2}n - \frac{3}{2} + \frac{2}{2k+4} + \frac{k(k+1)}{2k+2} + \frac{4}{k+4} + \frac{4k}{2k+3}$.

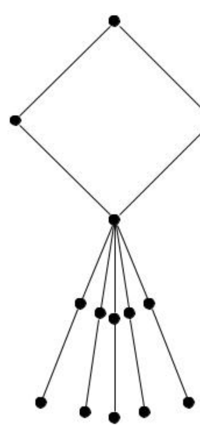

Figure 14: The graph of $C_4^5\{P_3\}$.

Now we will consider the graph $C_n^k\{P_{u+1}\}$ for u = 2. The graph $C_4^5\{P_3\}$ is shown in Figure **14**.

Proposition 3.2. Let G be a graph of $C_n^k\{P_3\}$, then

(1) $H_e(G; x) = 2(n-3)x^3 + 2kx^{k+2} + 4x^{k+3} + (k^2 + 3k + 2)x^{2k+3}$

(2) $H_e(G) = \frac{2k}{3+k} + \frac{4}{4+k} + \frac{1}{2}n - \frac{3}{2} + \frac{k^2+3k+2}{2k+4}$.

Proof. In the graph G, the total number of vertices and edges are $n + 2k$ and $n + 2k$ respectively (see Figure **14**). Therefore in line graph $L(G)$, the total number of vertices are $n + 2k$, out of which k vertices are of degree 1, n-2 vertices are of degree 2 and $k + 2$ vertices are of degree $k + 2$ (see Figure **15**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $\frac{k^2+5k+2n}{2}$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **6**.

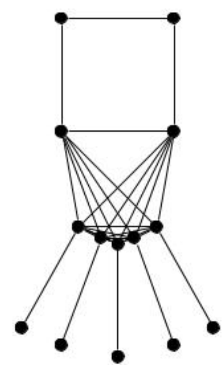

Figure 15: Line graph of $C_4^5\{P_3\}$.

Hence we get $H_e(G; x) = 2(n-3)x^3 + 2kx^{k+2} + 4x^{k+3} + (k^2 + 3k + 2)x^2$ and $H_e(G)$=

Now we will consider the graph $C_n^k\{P_{u+1}\}$ for $u > 2$. The graph $C_4^5\{P_4\}$ is shown in Figure **16**.

Proposition 3.3. Let G be a graph of $C_n^k\{P_{u+1}\}$, then

(1) $H_e(G, x) = (k^2 + 3k + 2)x^{2k+3} + 2(k + 2)x^{k+3} + 2\{n - 3 + (u-3)k\}x^3 + 2kx^2$;

Table 6: The Edge Partition of $L(G)$

$(d_e, d_f) \in E(L(G))$	(2, 2)	$(1, k+2)$	$(2; k+2)$	$(2, k+2)$	$(k+2, k+2)$
Number of edges	n - 3	k	2	2	$\frac{k^2+3k+2}{2}$

(2) $H_e(G)=\frac{2}{3}k+\left\{\frac{1}{2}n-\frac{3}{2}+\frac{1}{2}(u-3)k\right\}+\frac{k^2+3k+2}{2k+4}+\frac{2(k+2)}{k+4}$.

Proof. In the graph G, the total number of vertices and edges are $n + uk$ and $n + uk$ respectively (see Figure **16**). Therefore in line graph $L(G)$, the total number of vertices are $n + uk$, out of which k vertices are of degree 1, n-2 + (u-2)k vertices are of degree 2 and k + 2 vertices are of degree k + 2 (see Figure **17**). It is easily seen from Lemma 1.2 that the total number of edges in $L(G)$ are $\frac{k(k+1)+2(uk+n)}{2}$. The edge partition of $E(L(G))$ based on the degree of the vertices in shown in Table **7**.

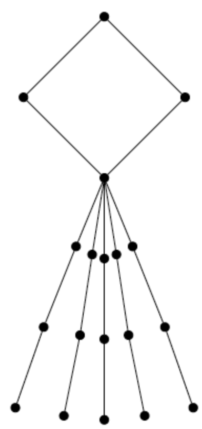

Figure 16: The graph of $C_4^5\{P_4\}$.

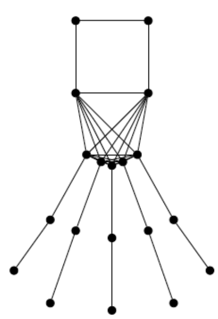

Figure 17: Line graph of $C_4^5\{P_4\}$.

Table 7: The Edge Partition of $L(G)$

$(d_e, d_f)2E(L(G))$	(1, 2)	(2,2)	(k + 2, k + 2)	(k + 2,2)
Number of edges	k	n- 3 + (u -3)k	$\frac{k^2+3k+2}{2}$	k + 2

Hence we get $H_e(G, x) = (k^2 + 3k + 2)x^{2k+3} + 2(k + 2)x^{k+3} + 2\{n - 3 + (u - 3)k\}x^3 + 2kx^2$ and $H_e(G)=\frac{2}{3}k+\left\{\frac{1}{2}n-\frac{3}{2}+\frac{1}{2}(u-3)k\right\}+\frac{k^2+3k+2}{2k+4}+\frac{2(k+2)}{k+4}$.

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