Solving the Periodic Toda-Type Chain with a Self-Consistent Source

Bazar Babajanov*, Azizbek Azamatov and Alisher Babajonov

Urgench State University, Urgench, 220100, Uzbekistan

Abstract: In this article, we explore the periodic Toda-type chain. The aim of this work is to obtain representations for the solutions of the periodic Toda-type chain with self-consistent source within the framework of the inverse spectral method for the discrete Hill equation. An efficient method for integrating the periodic Toda-type chain with self-consistent source is presented. The results can be used in modeling special types of electric transmission lines.

Keywords: Toda chain, discrete Hill Equation, self-consistent source, inverse spectral problem, trace formulas, soliton equations.

1. INTRODUCTION

The Toda chain [1] is a simple model for a nonlinear one-dimensional crystal that describes the motion of a chain of particles with exponential interactions of the nearest neighbors. The equation of motion for such a system is given by

$$\frac{d^2u_n}{dt^2} = \exp\left(u_{n-1} - u_n\right) - \exp\left(u_n - u_{n+1}\right),\ n \in Z,$$

where $u_n(t)$ the coordinate of the n th atom in a lattice. Using Flaschka's variables [2]

$$a_n = \frac{1}{2}\exp\left(\frac{u_n - u_{n+1}}{2}\right),\ b_n = \frac{1}{2}\dot{a}_n,$$

the Toda equation can be rewritten in the form

$$\begin{cases} \dot{a}_n = a_n(b_n - b_{n+1}), \\ \dot{b}_n = 2(a_{n-1}^2 - a_n^2), \quad n \in Z. \end{cases}$$

This equation has different practical applications. For example, the Toda lattice model of DNA in the field of biology [3]. Moreover, one important property of the Toda lattice type equations is the existence of so called soliton solutions. There is a close relation between the existence of soliton solutions and the integrability of equations: the known research results show that all the integrable systems have soliton solutions [4]. Soliton solutions of the Toda lattice are obtained in the works [2, 5]. Also, it is well known that the Toda lattice equation possesses rich families of solutions including rational solution, solitons, positons, negatons and soliton–positon, soliton–negaton, positon–negaton (see [6] for details). The periodic Toda lattice was considered in the works [7-10].

Here, we consider N-periodic Toda-type chain with self-consistent source

$$\begin{cases} \dot{a}_n = a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2) + a_n\sum\limits_{i=1}^{2N}\tilde{\theta}_{N+1}(\lambda_i,t)\left[(f_{n+1}^i)^2 - (f_n^i)^2\right] + \\ + a_n\int\limits_E \tilde{\theta}_{N+1}(\lambda,t)[\psi_{n+1}^-(\lambda,t)\psi_{n+1}^+(\lambda,t) - \psi_n^-(\lambda,t)\psi_n^+(\lambda,t)]d\lambda, \\ \dot{b}_n = 2a_n^2(b_{n+1} + b_n) - 2a_{n-1}^2(b_n + b_{n-1}) - 2\sum\limits_{i=1}^{2N}\tilde{\theta}_{N+1}(\lambda_i,t)f_n^i(a_n f_{n+1}^i - a_{n-1}f_{n-1}^i) + \\ + a_n\int\limits_E \tilde{\theta}_{N+1}(\lambda,t)[\psi_n^-(\lambda,t)\psi_{n+1}^+(\lambda,t) + \psi_{n+1}^-(\lambda,t)\psi_n^+(\lambda,t)]\,d\lambda - \\ - a_{n-1}\int\limits_E \tilde{\theta}_{N+1}(\lambda,t)[\psi_n^-(\lambda,t)\psi_{n-1}^+(\lambda,t) + \psi_{n-1}^-(\lambda,t)\psi_n^+(\lambda,t)]\,d\lambda, \\ a_{n-1}f_{n-1}^i + b_n f_n^i + a_n f_{n+1}^i = \lambda_i f_n^i, \\ a_{n+N} = a_n,\ b_{n+N} = b_n,\ (f_{n+N}^i)^2 = (f_n^i)^2,\ i = 1,2,\ldots,2N,\ a_n > 0, n \in Z,\ t \in R, \end{cases} \tag{1}$$

and the initial conditions

$$a_n(0) = a_n^0,\ b_n(0) = b_n^0,\ \ n \in Z, \tag{2}$$

with the given N-periodical sequences a_n^0 and b_n^0, $n \in Z$. In system (1), function sequences $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$, $\{f_n^1(t)\}_{-\infty}^{\infty}$, $\{f_n^2(t)\}_{-\infty}^{\infty}$, ..., $\{f_n^{2N}(t)\}_{-\infty}^{\infty}$, $\{\psi_n^{\pm}(\lambda,t)\}_{-\infty}^{\infty}$ – are unknown vector-functions, besides, $\{f_n^i(t)\}_{-\infty}^{\infty}$ and $\{\psi_n^{\pm}(\lambda,t)\}_{-\infty}^{\infty}$ are the Floquet-Bloch solutions for the discrete Hill's equation

$$\left(L(t)y\right)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{3}$$

normalized by conditions

$$\psi_1^{\pm}(\lambda,t) = 1,\ f_1^i(t) = 1,\ \ i = 1, 2, ..., 2N. \tag{4}$$

The eigenvalues λ_i of the Hill's equation are solutions of equation

$$\Delta^2(\lambda) - 4 = 0,$$

*Address correspondence to this author at the Urgench State University, Urgench, 220100, Uzbekistan; E-mail: a.murod@mail.ru

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where $\Delta(\lambda)=\theta_N(\lambda,t)+\phi_{N+1}(\lambda,t)$, and $\theta_n(\lambda,t)$, $n\in Z$ and $\phi_n(\lambda,t)$, $n\in Z$ are solutions of equation (3) under the initial conditions

$$\theta_0(\lambda,t)=1,\ \theta_1(\lambda,t)=0,\quad \phi_0(\lambda,t)=0,\ \phi_1(\lambda,t)=1.$$

In system (1), E is spectrum of the operator $L(0)$, and the factor $\tilde{\theta}_{N+1}(\lambda,t)$ is defined from the equality $\tilde{\theta}_{N+1}(\lambda,t)=\prod_{j=1}^{N-1}(\lambda-\mu_j(t))$, where $\mu_1(t),\mu_2(t),...,\mu_{N-1}(t)$ are the roots of the equation $\theta_{N+1}(\lambda,t)=0$.

Currently, the nonlinear evolution equations with self-consistent sources arouse active interest because of different physical applications. Usually, the right-hand side of nonlinear evolution equations with a self-consistent source integrable by the inverse spectral transform method consists of terms multiplied by integral factors depending on all the dynamical variables. They have important applications in plasma physics, hydrodynamics, solid-state physics, etc. [7-13]. For example, the KdV equation, which is included an integral type self-consistent source, was considered in [14]. By this type equation the interaction of long and short capillary-gravity waves can be described [15]. Other important soliton equations with self-consistent source are the nonlinear Schrodinger equation which describes the nonlinear interaction of an ion acoustic wave in the two component homogeneous plasma with the electrostatic high frequency wave [16]. Different techniques have been used to construct their solutions, such as inverse scattering [12, 13, 17, 18], Darboux transformation [20-23] or Hirota bi-linear methods [24–26]. Other aspects on integration of nonlinear periodical systems are presented in [27, 28, 29,30, 31, 32, 33, 34, 35].

The purpose of this paper consists on develop the scattering method for the periodic Toda-type chain equation with a self-consistent source. An effective method of integration of the Toda-type chain with a self-consistent source is presented.

The considered new system, similarly to [36, 37], can be used in some models of special types of electric transmission line.

2. THE BASIC INFORMATION ABOUT THE THEORY OF DIRECT AND INVERSE SPECTRAL PROBLEM FOR THE DISCRETE HILL'S EQUATION

In this section we give basic information about the theory of direct and inverse spectral problem for the discrete Hill's equation [1, 29].

We start with the following discrete Hill's equation

$$(Ly)_n \equiv a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n, \tag{5}$$

$$a_{n+N}=a_n,\ b_{n+N}=b_n,\ n\in Z,$$

with spectral parameter λ, and with period $N>0$. Let $\theta_n(\lambda)$, $n\in Z$ and $\phi_n(\lambda)$, $n\in Z$ be the solutions of equation (5) under the initial conditions

$$\theta_0(\lambda)=1,\ \theta_1(\lambda)=0,\ \phi_0(\lambda)=0,\ \phi_1(\lambda)=1.$$

Let $\lambda_1,\lambda_2,...,\lambda_{2N}$ be the roots of equation

$$\Delta^2(\lambda)-4=0.$$

We define the auxiliary spectrum $\mu_1,\mu_2,...,\mu_{N-1}$ as the roots of equation

$$\theta_{N+1}(\lambda)=0.$$

As it is known (see. [1]), all λ_i, $i=1,2,...,2N$ and $\mu_j, j=1,2,...,N-1$ are real, the roots μ_j are simple, but among the roots λ_i may occur the roots of multiplicity two.

It is easy to show, that

$$\Delta^2(\lambda)-4=\left(\prod_{j=1}^{N}a_j\right)^{-2}\prod_{j=1}^{2N}(\lambda-\lambda_j),$$

$$\theta_{N+1}(\lambda)=-a_0\left(\prod_{j=1}^{N}a_j\right)^{-1}\prod_{j=1}^{N-1}(\lambda-\mu_j).$$

We shall introduce

$$\sigma_j=sign\left[\theta_N(\mu_j)-\frac{1}{\theta_N(\mu_j)}\right],\ j=1,2,...,N-1.$$

Definition 1. The set of the numbers μ_j, $j=1,2,...,N-1$ and sequences of signs σ_j, $j=1,2,...,N-1$ is called spectral parameters of the discrete Hill's equation (5).

Definition 2. System of spectral parameters $\{\mu_j,\sigma_j\}_{j=1}^{N-1}$ and numbers λ_i, $i=1,2,...,2N$ is called spectral data of the discrete Hill's equation (5).

It is easy to see that the following statement is true.

Lemma 2. If $\{x_n(\lambda)\}_{-\infty}^{\infty}$ and $\{y_n(\mu)\}_{-\infty}^{\infty}$ are solutions of equations $Lx=\lambda x$ and $Ly=\mu y$, respectively. Then the identity

$$(\mu-\lambda)x_n(\lambda)y_n(\mu)=W\{x_n(\lambda),y_n(\mu)\}$$
$$-W\{x_{n-1}(\lambda),y_{n-1}(\mu)\},\quad n\in Z,$$

holds, where $W\{x_n(\lambda),y_n(\mu)\}=a_n[x_n(\lambda)y_{n+1}(\mu)-x_{n+1}(\lambda),y_n(\mu)]$.

3. EVOLUTION OF SPECTRAL PARAMETRS

In this section, we prove the basic result of this paper.

Theorem 1. If the functions $a_n(t)$, $b_n(t)$, $\{f_n^i(t)\}_{-\infty}^{\infty}$, $\psi_n^{\pm}(\lambda,t)$, $n\in Z$ are solutions of the problem (1)-(4), then the spectrum of discrete Hill operator (3) is independent of t, and spectral parametrs $\mu_j(t)$, $j=1,2,...,N-1$, satisfy the system of equations

$$\dot{\mu}_j(t)=-2\frac{\sigma_j(t)\cdot\sqrt{\prod_{k=1}^{2N}(\mu_j(t)-\lambda_k(t))}}{\prod_{\substack{k=1\\k\neq j}}^{N-1}(\mu_j(t)-\mu_k(t))}\cdot \left\{b_1(t)+\mu_j(t)+\sum_{i=1}^{2N}\frac{\tilde{\theta}_{N+1}(\lambda_i,t)}{\lambda_i-\mu_j(t)}+\int_E\frac{\tilde{\theta}_{N+1}(\lambda,t)}{\lambda-\mu_j(t)}d\lambda\right\}. \tag{6}$$

where

$$b_1(t)=\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{k=1}^{N-1}\left(\lambda_{2k}+\lambda_{2k+1}-2\mu_k(t)\right).$$

Proof. Let $y^j(t)=(y_0^j(t),y_1^j(t),...,y_N^j(t))^T$, $j=1,2,...,N-1$ denote the orthonormalized eigenvectors for the corresponding eigenvalues $\lambda=\mu_j(t)$, $j=1,2,...,N-1$, associated with the following boundary problem

$$\begin{cases}(L(t)y)_n\equiv a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n,\ 1\le n\le N\\ y_1=0,\ y_{N+1}=0.\end{cases}$$

In [11], was shown that

$$\dot{\mu}_j(t)=\sum_{n=1}^{N}(2\dot{a}_n(t)y_n^jy_{n+1}^j+\dot{b}_n(t)(y_n^j)^2).$$

Using (1), the last equality can be rewritten as follows

$$\dot{\mu}_j(t)=\sum_{n=1}^{N}2[a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2)]y_n^jy_{n+1}^j+$$
$$+\sum_{n=1}^{N}[2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})](y_n^j)^2+$$
$$+\sum_{n=1}^{N}\left\{\sum_{i=1}^{2N}[2a_n\tilde{\theta}_{N+1}(\lambda_i,t)((f_{n+1}^i)^2-(f_n^i)^2)y_n^jy_{n+1}^j]\right\}+$$
$$+\sum_{n=1}^{N}\left\{\sum_{i=1}^{2N}\left[2a_n\tilde{\theta}_{N+1}(\lambda_i,t)(f_n^if_{n+1}^i)(y_n^j)^2-2a_{n-1}\tilde{\theta}_{N+1}(\lambda_i,t)(f_{n-1}^if_n^i)(y_n^j)^2\right]\right\}+$$
$$+\sum_{n=1}^{N}\left\{2a_n\int_E\tilde{\theta}_{N+1}(\lambda,t)[\psi_{n+1}^-(\lambda,t)\psi_{n+1}^+(\lambda,t)-\psi_n^-(\lambda,t)\psi_n^+(\lambda,t)]d\lambda\right\}y_n^jy_{n+1}^j+$$
$$+\sum_{n=1}^{N}\left\{a_n\int_E\tilde{\theta}_{N+1}(\lambda,t)[\psi_n^-(\lambda,t)\psi_{n+1}^+(\lambda,t)+\psi_{n+1}^-(\lambda,t)\psi_n^+(\lambda,t)]d\lambda\right\}(y_n^j)^2-$$
$$-\sum_{n=1}^{N}\left\{a_{n-1}\int_E\tilde{\theta}_{N+1}(\lambda,t)[\psi_n^-(\lambda,t)\psi_{n-1}^+(\lambda,t)+\psi_{n-1}^-(\lambda,t)\psi_n^+(\lambda,t)]d\lambda\right\}(y_n^j)^2.$$

For convenience, let us put

$$G_i^j(\lambda,t)=\tilde{\theta}_{N+1}(\lambda_i,t)\sum_{n=1}^{N}\left\{2a_n[(f_{n+1}^j)^2-(f_n^j)^2]y_n^jy_{n+1}^j+\right.$$
$$\left.+2a_n(f_n^if_{n+1}^i)(y_n^j)^2-2a_{n-1}(f_{n-1}^if_n^i)(y_n^j)^2\right\},$$

$$F^j(\lambda,t)=\sum_{n=1}^{N}\left\{2a_n[\psi_{n+1}^-(\lambda,t)\psi_{n+1}^+(\lambda,t)-\psi_n^-(\lambda,t)\psi_n^+(\lambda,t)]\right\}y_n^jy_{n+1}^j+$$
$$+\sum_{n=1}^{N}\left\{a_n[\psi_n^-(\lambda,t)\psi_{n+1}^+(\lambda,t)+\psi_{n+1}^-(\lambda,t)\psi_n^+(\lambda,t)]\right\}(y_n^j)^2-$$
$$-\sum_{n=1}^{N}\left\{a_{n-1}[\psi_n^-(\lambda,t)\psi_{n-1}^+(\lambda,t)+\psi_{n-1}^-(\lambda,t)\psi_n^+(\lambda,t)]\right\}(y_n^j)^2,$$

$$H_n=2[a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2)]y_n^jy_{n+1}^j+$$
$$+[2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})](y_n^j)^2.$$

We will find sequences u_n, that $u_{n+1}-u_n=H_n$. We seek for u_n as follows

$$u_n=A_n(y_n^j)^2+2B_ny_n^jy_{n+1}^j+C_n(y_{n+1}^j)^2, \tag{7}$$

where $A_n=A_n(t,\mu_j), B_n=B_n(t,\mu_j)$ and $C_n=C_n(t,\mu_j)$ are unknown coefficients yet.

Due to

$$y_{n+2}^j=\frac{1}{a_{n+1}}[(\mu_j-b_{n+1})y_{n+1}^j-a_ny_n^j]$$

we have

$$\begin{gathered}(A_{n+1}-C_n)(y_{n+1}^j)^2-A_n(y_n^j)^2-2B_ny_n^jy_{n+1}^j\\+\frac{2B_{n+1}}{a_{n+1}}y_{n+1}^j[(\mu_j-b_{n+1})y_{n+1}^j-a_ny_n^j]+\\+\frac{C_{n+1}}{a_{n+1}^2}(\mu_j-b_{n+1})^2(y_{n+1}^j)^2-\frac{2C_{n+1}}{a_{n+1}^2}\\a_n(\mu_j-b_{n+1})y_n^jy_{n+1}^j+\frac{C_{n+1}}{a_{n+1}^2}a_n^2(y_n^j)^2=H_n\end{gathered}. \tag{8}$$

From the equality (8) we get

$$-B_n-\frac{a_n}{a_{n+1}}B_{n+1}-\frac{a_n(\mu_j-b_{n+1})}{a_{n+1}^2}C_{n+1}=a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2), \tag{9}$$

$$\begin{gathered}-C_{n-1}+\frac{2(\mu_j-b_n)}{a_n}B_n+\frac{(\mu_j-b_n)^2}{a_n^2}C_n+\frac{a_n^2}{a_{n+1}^2}C_{n+1}\\=2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})\end{gathered}. \tag{10}$$

It is easy to check that

$$C_n=2a_n^2(\mu_j+b_n),\ B_n=a_n(a_{n-1}^2-a_n^2+b_n^2-\mu_j^2)$$

are solutions to the system (9) and (10). By virtue of (7), we obtain

$$\dot\mu_j(t)=\sum_{n=1}^N 2[a_n(a_{n+1}^2-a_{n-1}^2)+a_n(b_{n+1}^2-b_n^2)]y_n^jy_{n+1}^j+$$

$$+\sum_{n=1}^N[2a_n^2(b_{n+1}+b_n)-2a_{n-1}^2(b_n+b_{n-1})](y_n^j)^2$$

$$+\sum_{i=1}^{2N}G_i^j(\lambda,t)+\int_E\tilde\theta_{N+1}(\lambda,t)F^j(\lambda,t)\,d\lambda=$$

$$=C_{N+1}(y_{N+2}^j)^2-C_1(y_2^j)^2+\sum_{i=1}^{2N}G_i^j(\lambda,t)$$

$$+\int_E\tilde\theta_{N+1}(\lambda,t)F^j(\lambda,t)\,d\lambda=$$

$$\begin{gathered}=2a_0^2(\mu_j(t)+b_1(t))[(y_N^j)^2-(y_0^j)^2]\\+\sum_{i=1}^{2N}G_i^j(\lambda,t)+\int_E\tilde\theta_{N+1}(\lambda,t)F^j(\lambda,t)\,d\lambda\end{gathered}. \tag{11}$$

Using the form of $G_i^j(\lambda,t)$ and $F^j(\lambda,t)$, we find that

$$G_i^j(\lambda,t)=\tilde\theta_{N+1}(\lambda_i,t)$$

$$\sum_{n=1}^N\left[2a_nf_{n+1}^iy_{n+1}^j(y_n^jf_{n+1}^i-y_{n+1}^jf_n^i)+2a_nf_n^iy_n^j(y_n^jf_{n+1}^i-f_n^iy_{n+1}^j)\right]=$$

$$=\tilde\theta_{N+1}(\lambda_i,t)\left[2\sum_{n=1}^N f_{n+1}^iy_{n+1}^jT_n+2\sum_{n=0}^N f_{n+1}^iy_{n+1}^jT_{n+1}\right]$$

$$=\tilde\theta_{N+1}(\lambda_i,t)\left[2\sum_{n=1}^N f_{n+1}^iy_{n+1}^j(T_n+T_{n+1})\right]=$$

$$\begin{gathered}=2\tilde\theta_{N+1}(\lambda_i,t)\sum_{n=1}^N\frac{1}{\lambda_i-\mu_j(t)}(T_{n+1}-T_n)(T_{n+1}+T_n)\\=\frac{2\tilde\theta_{N+1}(\lambda_i,t)}{\lambda_i-\mu_j(t)}(T_{N+1}^2-T_1^2)\end{gathered},$$

where $T_n=a_n(y_n^jf_{n+1}^i-y_{n+1}^jf_n^i)$. Thus, we get

$$\begin{gathered}G_i^j(\lambda,t)=\frac{2\tilde\theta_{N+1}(\lambda_i,t)a_0^2(f_1^i)^2}{\lambda_i-\mu_j(t)}[(y_N^j)^2-(y_0^j)^2],\\F^j(\lambda,t)=\frac{2a_0^2}{\lambda-\mu_j(t)}[(y_N^j)^2-(y_0^j)^2].\end{gathered} \tag{12}$$

Substituting (12) in (11), we derive

$$\begin{gathered}\dot\mu_j(t)=2a_0^2[(y_N^j)^2-(y_0^j)^2]\\\left\{\mu_j(t)+b_1(t)+\sum_{i=1}^{2N}\frac{\tilde\theta_{N+1}(\lambda_i,t)}{\lambda_i-\mu_j(t)}+\int_E\frac{\tilde\theta_{N+1}(\lambda,t)}{\lambda-\mu_j(t)}d\lambda\right\}\end{gathered}. \tag{13}$$

By virtue of the equalities

$$\left\|\theta^j\right\|^2=\sum_{n=1}^N(\theta_n^j)^2=a_N\theta_N^j(\theta_{N+1}^j)'\big|_{\lambda=\mu_j},\ (\theta^j)'=\frac{d\theta^j}{d\lambda},$$

$$(y_0^j)^2=\frac{(\theta_0^j)^2}{\left\|\theta^j\right\|^2},\ (y_N^j)^2=\frac{(\theta_N^j)^2}{\left\|\theta^j\right\|^2},$$

we can write the equation (13) in the form

$$\begin{gathered}\dot\mu_j(t)=\frac{2a_0\left(\theta_N^j(\mu_j(t),t)-\dfrac{1}{\theta_N^j(\mu_j(t),t)}\right)}{(\theta_{N+1}^j)'\big|_{\lambda=\mu_j(t)}}\cdot\\\left\{\mu_j(t)+b_1(t)+\sum_{i=1}^{2N}\frac{\tilde\theta_{N+1}(\lambda_i,t)}{\lambda_i-\mu_j(t)}+\int_E\frac{\tilde\theta_{N+1}(\lambda,t)}{\lambda-\mu_j(t)}d\lambda\right\}\end{gathered}. \tag{14}$$

It is easy to check that

$$\theta_N^j(\mu_j(t),t)-\frac{1}{\theta_N^j(\mu_j(t),t)}=\sigma_j(t)\sqrt{\Delta^2(\mu_j(t))-4}, \tag{15}$$

$$\theta'_{N+1}(\lambda)\big|_{\lambda=\mu_j(t)}=-a_0\left(\prod_{k=1}^N a_k\right)^{-1}\prod_{\substack{k=1\\k\neq j}}^{N-1}(\mu_j(t)-\mu_k(t)), \tag{16}$$

where $\sigma_j(t) = sign\left(\theta_N^j(\mu_j(t),t) - \phi_{N+1}^j(\mu_j(t),t)\right)$, $j = 1, 2, ..., N-1$.

Substituting (15) and (16) in (14) we obtain equality (6).

We now show that $\lambda_k(t)$ is independent of t. Let $\left\{g_n^k(t)\right\}$ be the normalized eigenfunction of the operator $L(t)$ corresponding to the eigenvalue $\lambda_k(t)$, $k = 1, 2, ..., 2N$, i.e.

$$a_{n-1}g_{n-1}^k + b_n g_n^k + a_n g_{n+1}^k = \lambda_k g_n^k .$$

By differentiating the last identity with respect to t, multiplying by g_n^k and summing over n we get

$$\frac{d\lambda_k}{dt} = \sum_{n=1}^{N}\left(2\dot{a}_n(t)g_n^k g_{n+1}^k + \dot{b}_n(t)\left(g_n^k\right)^2\right). \tag{17}$$

Using the equation (1), we can write the equality (17) as

$$\begin{aligned}\dot{\lambda}_k(t) &= 2\sum_{n=1}^{N}[a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2)]g_n^k g_{n+1}^k + \\ &+\sum_{n=1}^{N}[2a_n^2(b_{n+1}+b_n) - 2a_{n-1}^2(b_n + b_{n-1})](g_n^k)^2 \\ &+\sum_{i=1}^{2N} G_i^j(\lambda,t) + \int_E \tilde{\theta}_{N+1}(\lambda,t)F^j(\lambda,t)\,d\lambda\end{aligned} \tag{18}$$

Similarly to (13), from the equality (18) we get $\dot{\lambda}_k(t) = 0$. **The theorem is proved.**

Corollary. *If $N = 2p$ and the number p is the period of the initial sequences $\{a_n^0\}$ and $\{b_n^0\}$, then all roots of the equation $\Delta(\lambda) + 2 = 0$ are double roots. Because the Lyapunov function corresponding to the coefficients $a_n(t)$ and $b_n(t)$ coincides with $\Delta(\lambda)$, according to the analogue of the Borg inverse theorem for the discrete Hill equation (see* [38]*), the number p is also the period of the solution $a_n(t)$, $b_n(t)$ with respect to the variable n.*

4. CONCLUSION

Theorem 1 provides the method for solving the problem (1)-(4).

(i) Solving the direct spectral problem for the discrete Hill's equation with $\{a_n^0\}$ and $\{b_n^0\}$ the spectral data $\lambda_i,\ i = 1,2,...,2N$ and $\mu_j(0),\ \sigma_j(0),\ j = 1,2,...,N-1$ are obtained.

(ii) Using the result of Theorem 1, we find the $\mu_j(t),\ \sigma_j(t),\ j = 1,2,...,N-1$

(iii) Using the algorithm which is presented in [29], we calculate $a_n(t)$, $b_n(t)$ and hence $\{f_n^i(t)\}_{-\infty}^{\infty}$, $\psi_n^{\pm}(\lambda,t)$.

5. EXAMPLE

Let us illustrate the application of the main theorem for solving problem (1) - (2) with the initial conditions

$$(a_n^0)^2 = \frac{5}{2} - (-1)^n\frac{3}{2},\ b_n^0 = 0,\ n \in Z.$$

In this case,

$$N = 2,\ \lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 1,\ \lambda_4 = 3,$$
$$\mu_1(0) = 0,\ \sigma_1(0) = 1.$$

Using steps of conclusion, we obtain

$$a_n^2(t) = \frac{5}{2} - \frac{1}{2}\mu^2(t) - (-1)^n\frac{\sigma(t)}{2}\sqrt{(\mu^2(t)-1)(\mu^2(t)-9)},$$

$$b_n(t) = (-1)^n\mu(t),\ n \in Z,$$

$$f_0^k(t) = \frac{\lambda_k^2 - \mu^2(t) - \sigma(t)\sqrt{(\mu^2(t)-9)(\mu^2(t)-1)}}{2a_0(t)(\lambda_k - \mu(t))},\ ,$$
$$f_1^k(t) = 1,\ k = 1,2,3,4$$

where $\mu(t)$ can be determined from the equation

$$\frac{d\mu(t)}{dt} = -12\sigma(t)\sqrt{(1-\mu^2(t))(9-\mu^2(t))},$$

with the initial conditions $\mu(0) = 0$, $\sigma(0) = 1$, and the function $\sigma(t)$ changes sign in each collision of the point $\mu(t)$ with the ends of the gap $[-1, 1]$. Introducing the transformation $\mu(t) = \sin x(t)$, and using the equality

$$sign\ \sigma(t)\cdot sign\ (\cos x(t)) = \sigma(0),$$

we obtain

$$a_n^2(t) = \frac{5}{2} - \frac{1}{2}\sin^2 x(t) - (-1)^n\frac{3}{2}\cos x(t)\sqrt{1-\left(\frac{1}{3}\right)^2\sin^2 x(t)},$$

$$b_n(t) = (-1)^n\sin x(t),\ n \in Z,$$

$$f_0^k(t) = \frac{\lambda_k^2 - \sin^2 x(t) - \cos x(t)\sqrt{9-\sin^2 x(t)}}{2a_0(t)(\lambda_k - \sin x(t))},\ ,$$
$$f_1^k(t) = 1,\ k = 1,2,3,4$$

where $x(t)$ is the solution of the Cauchy problem

$$\dot{x}(t) = -36\sqrt{1-\left(\frac{1}{3}\right)^2 \sin^2 x(t)} \,,$$

$$x(0) = 0 \,.$$

It is known that (см. [39])

$$x(t) = am\left(-36t, \frac{1}{3}\right),$$

here am is the Jacobi amplitude function. Therefore,

$$a_n(t) = \sqrt{\frac{5}{2} - \frac{1}{2} sn^2\left(-36t, \frac{1}{3}\right) - (-1)^n \frac{3}{2} cn\left(-36t, \frac{1}{3}\right) dn\left(-36t, \frac{1}{3}\right)},$$

$$b_n(t) = (-1)^n sn\left(-36t, \frac{1}{3}\right),$$

$$f_0^k(t) = \frac{\lambda_k^2 - sn^2\left(-36t, \frac{1}{3}\right) - 3cn\left(-36t, \frac{1}{3}\right) dn\left(-36t, \frac{1}{3}\right)}{2a_0(t)\left[\lambda_k - sn\left(-36t, \frac{1}{3}\right)\right]},$$

$$f_1^k(t) = 1, \; k = 1,2,3,4,$$

where sn, cn and dn are the Jacobi elliptic functions.

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