On the Spectral Expansions Connected with Schrödinger's Operator of Continuous Functions in a Closed Domain

Abdumalik A. Rakhimov[1,2,*], Kamran Zakaria[2,3] and Nazir Ali Khan[3]

[1]*Department of Science in Engineering, International Islamic University Malaysia, Malaysia*

[2]*Laboratory of Computational Sciences and Mathematical Physics, INSPEM, UPM, Malaysia*

[3]*Math Sciences Research Center, Federal Urdu University, Karachi, Pakistan*

Abstract: Present paper is devoted to study of uniformly convergence of spectral expansions in a closed domain. We consider here as a spectral expansions eigenfunction expansions connected with one Schrodinger's operator with singular potential in two dimensional domains with smooth boundary.

Keywords: Eigenfunction, Eigenvalues, Expansions, Regularization, Riesz means, Mean value formula, Singular potential, Boundary problem, Hamiltonian, Sobolev spaces.

INTRODUCTION

One of very important operators in quantum mechanics is Schrödinger's operator with singular potential. This operator is acting on the functions belonging to the Hilbert space [1]. As a domain of this operator we can consider appropriate the Sobolev spaces [2, 3] and domain must be such that the operator must be self-adjoint [3]. For self-adjoint operator the eigenvalues are real and the eigenfunctions form a complete set of orthogonal functions so that any function of the Hilbert space of the system can be expanded in this set in strong topology. If Schrödinger's operator is "good' perturbation of Laplace operator then corresponding extension can be self-adjoint operator. For example, for the potentials from $W_2^1(\Omega)$ first boundary value problem in smooth domain is self-adjoint [4]. When the Hamiltonian H of the system is a self-adjoint operator, then the solution of the time-dependent Schrodinger equation can be presented by spectral expansions connected with corresponding Hamiltonian. In its term finding this solution leads to the investigation of convergence and summability problems, related to the eigenfunction expansion in a closed domain. Methods of study of spectral expansions in compact subsets of the domain is well developed and known (see in [2]). But n closed domains some difficulties occur near the boundary. These difficulties can be avoided if we consider boundary conditions that help to estimate eigenfunctions expansions in the closed domain. study convergence and/or summability of such spectral expansions. In case of free Hamiltonian corresponding methods and theory developed by many scientists (see in [2, 5]).

In present paper we study uniformly convergence of regularized eigenfunction expansions connected with Schrodinger's operator in a class of continuous functions. This problem for the free Hamiltonian operator was studied by A. A. Rakhimov in [6] and in case of spectral expansions connected with arbitrary elliptic operator with smooth coefficient by Sh.A. Alimov and A.A. Rakhimov in [7, 8]. Eigenfunction expansions connected with Schrodinger's operator in compact subsets of the domain studied by A.R. Khalmukhamedov [9]. Estimation of the eigenfunctions for free Hamiltonian was obtained by E.I. Moiseev in [10] and he obtained uniformly convergence in the Sobolev spaces.

Definitions and Formulations of the Results

Let Ω a bounded domain in R^2 with smooth boundary $\partial\Omega$. We will consider potential function *q(x)* as a positive function from Sobolev's space $W_2^1(\Omega)$ with singularities at a point $x_0 \in \Omega$ (or in finite numbers of points) and enough smooth out of this point, so we can suppose that all volume improper integrals below which contains this function and its partial derivatives are exist and finite. Also we will suppose that first eigenvalue problem for the corresponding Schrödinger's operator with potential q produce selfadjoint operator so the problem has countable number Eigenfunctions $u_n(x)$ answering to the eigenvalues λ_n. First, we will consider estimations for the solutions of the following boundary value problem:

*Address corresponding to this author at the Department of Science in Engineering, International Islamic University Malaysia; Tel: (+603) 6196 6523; Fax: (+603) 6196 4053; E-mails: abdumalik@iium.edu.my, or abdumalik2004@mail.ru

ISSN: 1814-8085 / E-ISSN: 1927-5129/12

$$\Delta u + qu + \mu^2 u = f \tag{1}$$

$$u|_{\partial\Omega} = 0 \tag{2}$$

where $f \in L_2$, Δ is Laplace operator.

Following estimation for the solution of the problem (1)-(2) in a closed domain $\overline{\Omega}$ is important for this paper.

Lemma 1: For any solution of the problem (1)-(2) following estimation is valid.

$$\|u\|_{c(\overline{\Omega})} \leq \sqrt{\frac{\ln^2 \mu_0}{\mu_0}} \|f\|_{L_2(\Omega)}, \quad \mu_0 = \operatorname{Re}\mu, \tag{3}$$

where $\mu_0 \to \infty$.

Proof of this statement can be found in [7].

The Riesz means of nonnegative order s of the partial sums of eigenfunction expansions by system $\{u_n(x)\}$ defined by following equality

$$E_\lambda^s f(x) = \sum_{\lambda_n \leq \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x), \tag{4}$$

where $f_n = (f, u_n)$ and $\{\lambda_n\}$- a sequence of eigenvalues: $0 < \lambda_1 \leq \lambda_2 \leq \ldots\ldots\ldots\ldots\ldots \leq \lambda_n \to \infty$.

We prove following theorem

__Theorem.__ Let f a finite and continuous in a domain G function. Then Reisz means (1) of order s, $s > \frac{N-1}{2}$, convergence to f uniformly in a closed domain $\overline{G}$.

Preliminaries and Auxiliary Lemmas

It is important to obtain appropriate representation of the partial sums of eigenfunction expansion. This representation can be obtained by standard way (see for example in [4, 8]) using mean value formula for the solution of th equation. For any number $h > 0$ by G_h denote following set $G_h = \{x \in G : dist(x, \partial G) > h\}$.

Let $x \in G_h$ и $y \in \overline{G}$. Consider following well-known function of variable $r = |x - y|$:

$$V(r) = \begin{cases} \Gamma(s+1) 2^s (2\pi)^{\frac{-N}{2}} \lambda^{\frac{N}{4} - \frac{s}{2}} \dfrac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{r^{\frac{n}{2}+s}}, & r \leq R \\ 0, & R > 0 \end{cases}, \tag{5}$$

where R less $\frac{h}{4}$, $J_\nu(t)$– Bessel's function of order ν.

For eigenfunctions $u_n(x)$ we have following mean value formula in a ball $\{ \leq R\}$ with the centre at $x \in G_h$:

$$S_t(u_n) = (2\pi)^{N/2} J_\beta(r\sqrt{\lambda_n})(r\sqrt{\lambda_n})^{-\beta} u_n(x) + \tag{6}$$
$$+\frac{\pi}{2} r^{-\beta} \int_0^r \left\{J_\beta(t\sqrt{\lambda_n}) Y_\beta(r\sqrt{\lambda_n}) - Y_\beta(t\sqrt{\lambda_n}) J_\beta(r\sqrt{\lambda_n})\right\} t^{\beta+1} S_t(qu_n) dt$$

where $\beta = \frac{N-2}{2}$, $S_t(g)(x) = \int_\theta g(x + t\theta) d\theta$, and $J_\nu(t)$, $Y_\nu(t)$– are Bessel's functions of order ν.

Note that

$$\int_0^\infty J_{a+s}(\sqrt{\lambda}\, t) J_{a-1}(\sqrt{\lambda_k}\, t) t^{-s} dt = \begin{cases} \dfrac{\left(1 - \frac{\lambda_k}{\lambda}\right)^s \lambda^s \lambda_k^{\frac{a-1}{2}}}{2^s \Gamma(s+1) \lambda^{\frac{a+s}{2}}}, & \lambda_k \leq \lambda \\ 0, & \lambda_k > \lambda \end{cases} \tag{7}$$

Using (6) we obtain following expression for Fourier coefficient of function $\overset{\lambda}{v}(|x-y|)$:

$$\overset{\lambda}{v}_n(x) = 2^s \Gamma(s+1) \lambda_n^{\frac{2-N}{4}} \lambda^{\frac{N-2s}{4}} u_n(x) \cdot \int_0^R J_{\frac{N}{2}+s}(\sqrt{\lambda} r) J_{\frac{N}{2}-1}(\sqrt{\lambda_n} r) r^{-s} dr + \tag{8}$$
$$+ \frac{2^s \Gamma(s+1)}{(2\pi)^{N/2}} \frac{\pi}{2} \lambda^{\frac{N}{2}} \cdot \int_0^R \left(r\sqrt{\lambda}\right)^{-\frac{N}{2}-s} J_{\frac{N}{2}+s}\left(r\sqrt{\lambda}\right) \cdot r^{N-1-\beta} \cdot$$
$$\cdot \int_0^r W_\beta(t, r, \sqrt{\lambda_n}) \cdot t^{\beta+1} \cdot S_t(q \cdot u_n) dt$$

where $W_\beta(t, r, \sqrt{\lambda_n}) = J_\beta(t\sqrt{\lambda_n}) Y_\beta(r\sqrt{\lambda_n}) - Y_\beta(t\sqrt{\lambda_n}) J_\beta(r\sqrt{\lambda_n})$.

In right side of (8) divide first integral into two part as $\int_0^\infty - \int_R^\infty$ and taking into consideration (7) obtain following formula for $\overset{\lambda}{v}_n(x)$:

$$\overset{\lambda}{v}_n(x) = \delta_n^\lambda u_n(x)(1-\frac{\lambda_n}{\lambda})^s - $$
$$- \quad 2^s\Gamma(s+1)\lambda_n^{\frac{1-N}{4}}\lambda^{\frac{N-1}{4}-\frac{s}{2}}u_n(x)I_1(\lambda,\lambda_n) + $$
$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot I_2(\lambda,u_n), \tag{9}$$

where

$$I_1(\lambda,\lambda_n) = (\lambda\cdot\lambda_n)^{1/4}\int_R^\infty J_{\frac{N}{2}+s}(r\sqrt{\lambda})J_{\frac{N}{2}-1}(r\sqrt{\lambda_n})r^{-s}dr$$

$$\delta_n^\lambda = \begin{cases} 1, & \lambda_n < \lambda \\ 0, & \lambda_n \ge \lambda \end{cases},$$

and

$$I_2(\lambda,u_n) = \cdot\int_0^R \left(r\sqrt{\lambda}\right)^{-\frac{N}{2}-s} J_{\frac{N}{2}+s}\left(r\sqrt{\lambda}\right)\cdot r^{N-1-\beta}\cdot \int_0^r W_\beta(t,r,\sqrt{\lambda_n})\cdot t^{\beta+1}\cdot S_t(q\cdot u_n)dt \ , \tag{10}$$

Multiply both side of (9) to $u_n(x)$ and take summation by all numbers. As a result obtain following equality in sense of L_2 by y :

$$\overset{\lambda}{v}\left(|x-y|\right) = \Theta^s(x,y,\lambda) -$$
$$-2^s\Gamma(s+1)\lambda^{\frac{N-1}{4}-\frac{s}{2}}\sum_{n=1}^\infty u_n(x)\cdot u_n(y)\cdot\lambda_n^{\frac{1-n}{4}}\cdot I_1(\lambda,\lambda_n) \ , \tag{11}$$
$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot\sum_{n=1}^\infty u_n(y)\cdot I_2(\lambda,u_n)$$

where

$$\Theta^s(x,y,\lambda) = \sum_{\lambda_n<\lambda}\left(1-\frac{\lambda_n}{\lambda}\right)^s\cdot u_n(x)\cdot u_n(y)$$

is called Reisz means of spectral function.

Left side of (11) in $x\in G_h$ and $y\in\overline{G}$, denote by $V(x,y,\lambda)$.

Lemma 2. Following uniformly by $y\in\overline{G}$ estimation for eigenfunctions of first boundary problem for Schrodinger's operator is true:

$$\sum_{\left|\sqrt{\lambda_n}-\mu\right|\le 1} u_n^2(y) = O(\mu\ln^2\mu) \tag{12}$$

Proof. Let $R(x,y,\mu)$ is the resolvent operator of the problem (1)-(2), so any solution $u(x)$ of the problem can be represented as

$$u(x) = \int_\Omega R(x,y,\mu)f(y)dy$$

From Lemma 1 it follows the uniformly on $x\in\overline{\Omega}$ one has

$$\|R(x,y,\mu)\|_{L_2(\Omega)} \le \frac{\ln\mu_0}{\sqrt{\mu_0}}, \text{ R(x,y,μ).-,L-2.(Ω).≤} \tag{13}$$

as $\mu_0\to\infty$

From the other hand it follows that for any eigenfunction ,u-k.(x) $u_k(x)$ we have

$$u_k(x) = (\mu^2-\lambda_k)\int_\Omega R(x,y,\mu)f(y)dy$$

Thus using Parseval inequality, we obtain

$$\left|\sum_{\left|\sqrt{\lambda_n}-\mu_0\right|\le 1} R(x,y,\mu)f(y)\right| \le \mu_0^2\,\|R(x,y,\mu)\|_{L_2(\Omega)}^2 \tag{14}$$

Then statement of the lemma 2 immediately follows from (13). Lemma 2 is proved.

Then from (12) it follows that for any positive number ε it is valid estimation:

$$\sum_{\lambda_n<\lambda} u_n^2(y)\cdot\lambda_n^{\varepsilon-1} = O(\lambda^\varepsilon\cdot\ln^2\lambda) \tag{15}$$

$$\sum_{\lambda_n>\lambda} u_n^2(y)\lambda_n^{-\varepsilon-1} = O(\lambda^{-\varepsilon}\cdot\ln^2\lambda) \tag{16}$$

For integral $I_1(\lambda,\lambda_n)$ defined we have following estimation (see in [5]):

$$|I_1(\lambda,\lambda_n)| \quad \le \quad \frac{c}{1+\left|\sqrt{\lambda_n}-\sqrt{\lambda}\right|} \tag{17}$$

Let $f\in L_2(G)$ and f_n its Fourier coefficients. Then from estimations (15) and (16) it follows that series

$$\sum_{n=1}^\infty f_n\cdot u_n(y)\cdot\lambda_n^{\frac{-1}{2}}\cdot I_1(\lambda,\lambda_n),$$

convergence uniformly in closed domain $\overline{G}$. Therefore, for any function f from $L_2(G)$ integral

$$\int_G f(x)\quad V(x,y,\lambda)\quad dx$$

is continuous by $y\in\overline{G}$.

Let's support of a function $f(x)\in L_2(G)$ is in G_h. Then by definition for Reisz means of partial sum of Fourier series of function $f(x)$ by system $\{u_n(x)\}$, we will have

$$E_\lambda^S f(y) = \int_{G_h} f(x)\ V(x,y,\lambda)\ dx +$$

$$+\ 2^S \text{n}\,(s+1)\lambda^{\frac{N-1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} f_n \lambda_n^{\frac{1-N}{4}}\ u_n(y)\ I_1(\lambda,\lambda_n) + \tag{18}$$

$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot\sum_{n=1}^{\infty} f_n I_2(\lambda,u_n)$$

Denote by $B(R,y)$ a ball of radius R with the center at point $y\in\overline{G}$. Then taking into account continuity of function

$$\int_{G_h} f(x)\ \overset{\lambda}{v}(r)\ dx = 2^s\text{n}\,(s+1)\ (2\pi)^{-\frac{N}{2}}\ \lambda^{\frac{N}{4}-\frac{s}{2}}\times$$

$$\times\int_{G_n \text{I}\ B(R,y)} f(x)J_{\frac{N}{2}+s}(\sqrt{\lambda}r)\cdot r^{-(\frac{N}{2}+s)}\ dr$$

and also fact that it is equal to first term in first part equality (13), obtain

$$E_\lambda^s f(y) = \int_{G_h} f(x)\overset{\lambda}{v}(|x-y|)dx +$$

$$+2^s\cdot\text{n}\,(s+1)\cdot\lambda^{\frac{N-1}{4}-\frac{s}{2}}\cdot\sum_{n=1}^{\infty} f_n\lambda_n^{\frac{1-N}{4}}\cdot u_n(y)I_1(\lambda,\lambda_n).\ + \tag{19}$$

$$+\frac{2^s\Gamma(s+1)}{(2\pi)^{N/2}}\frac{\pi}{2}\lambda^{\frac{N}{2}}\cdot\sum_{n=1}^{\infty} f_n I_2(\lambda,u_n)$$

Lemma 3. let $y\in\overline{G}$. Then uniformly by y we have unequality

$$\sum_{n=1}^{\infty} u_n^2(y)\ \lambda_n^{\frac{1-N}{2}}\ \ [I_1(\lambda,\lambda_n)]^2 \ \le\ C\cdot\text{l}\,n^2\lambda\ . \tag{20}$$

Proof. From (15) and (17) taking as $\lambda=1$ it follows

$$\sum_{\sqrt{\lambda_n}\le 1} \lambda_n^{\frac{1-N}{2}}\ \left[I_1(\lambda,\lambda_n)\right]^2\ u_n^2(y)\ =\ O\left(\frac{1}{\lambda}\right).$$

Also from (17) , (16) and (15) obtain following estimations:

$$\sum_{1\le\sqrt{\lambda_n}\le\frac{\sqrt{\lambda}}{2}} \lambda_n^{\frac{1-N}{2}}\left[I_1\left(\lambda,\lambda_n\right)\right]^2 u_n^2(y) = O\left(\frac{\text{l}\,n^2\lambda}{\sqrt{\lambda}}\right) \tag{21}$$

$$\sum_{\frac{\sqrt{3\lambda}}{2}\le\sqrt{\lambda_n}} \lambda_n^{\frac{1-N}{2}}\left[I_1\left(\lambda,\lambda_n\right)\right]^2 u_n^2(y) = O\left(\frac{\text{l}\,n^2\lambda}{\sqrt{\lambda}}\right)\ . \tag{22}$$

For estimating term which answer to the numbers n for which $\frac{\sqrt{\lambda}}{2}\le\sqrt{\lambda_n}\le\frac{\sqrt{3\lambda}}{2}$ we use (14) and (17). Denote by k least number for which $2^x\ \ge\ \frac{\sqrt{\lambda}}{2}$. Then taking into account (14) and (17) obtain

$$\sum_{\left|\sqrt{\lambda_n}-\sqrt{\lambda}\right|\le\frac{\sqrt{\lambda}}{2}} \lambda_n^{\frac{1-N}{2}}\left[I_1\left(\lambda,\lambda_n\right)\right]^2 u_n^2(y)\le$$

$$\sum_{m=1}^{k}\ \sum_{2^{m-1}\le\left|\sqrt{\lambda}-\sqrt{\lambda_n}\right|\le 2^m} \lambda_n^{\frac{1-N}{2}} u_n^2(y)\,4^{1-m}\le c\cdot\text{l}\,n^2\lambda\ .$$

Lemma 3 proved.

Lemma 4. Let function $f(y)$ continuous and finite in G. If $s>(N-1)/2$, the uniformly by $y\in\overline{G}$ following inequality is valid

$$\left|E_\lambda^s f(y)\right|\ \ \le\ \ c\ \|f\|_\infty\ . \tag{23}$$

Proof. First estimate each of terms in right side of (19). For estimation of first term we use following estimations for Bessel's function

$$\left|J_\nu(t)\right|\ \ \le\ \ \begin{cases} t^{-1/2}\ , & t\ge 1\\ t^{\nu}\ , & t\le 1\end{cases}.$$

Then dividing integral in right side of (19) into two obtain

$$\left|\int_{G_n} f(x)\cdot\overset{\lambda}{v}(r)\,dx\right|\le c_1\cdot\|f\|_\infty$$

$$\left[\int_0^{1/\sqrt{\lambda}} r^{N-1}\cdot\left|\overset{\lambda}{v}(r)\right|dx+\int_{1/\sqrt{\lambda}}^{R} r^{N-1}\cdot\left|\overset{\lambda}{v}(r)\right|dr\right].$$

It is clear that from estimation of Bessel's function it follows that quantity in quadratic brackets bounded. Now estimate second term in right side of equality (19). For this we apply Holder's inequality and then Parsevvall's equality to the following sum

$$\sum_{n=1}^{\infty} f_n\cdot u_n(y)\cdot I_1(\lambda,\lambda_n)\ \ \lambda_n^{\frac{1-N}{4}}$$

Then from lemma 3 it follows

$$\left|\sum_{n=1}^{\infty} f_n\cdot u_n(y)\cdot I_1\left(\lambda,\lambda_n\right)\cdot\lambda_n^{\frac{1-N}{4}}\right|\ \ \le\ \ c\cdot\text{l}\,n\lambda\ \ \|f\|_\infty$$

Third term in left side of (19) can be proved as second term. Lemma 4 is proved.

Proof of the Theorem

For any number $s \geq 0$ convergence of $E_\lambda^s f(y)$ is always uniformly in closed domain when a function belongs to the space $C_0^\infty(G)$. Note that space of smooth functions with compact support is dense in the space of continuous function on G. Thus a function $f(y)$ which satisfies conditions of the theorem can be approximated by functions from $C_0^\infty(G)$, which has supports in G_h, where positive number *h* depends only from distance between support of the function $f(y)$ and boundary of the domain G. Then statement of the theorem follows from lemma 4. Theorem is proved.

REMARKS

Results of the paper can be used in investigations of the solvability of the problems in quantum mechanics, nuclear physics and mathematical physics.

ACKNOWLEDGMENTS

This work done with the support of IIUM University Reserch Grant (Type B) EDW B12-397-0875.

REFERENCES

[1] Reed M., Simon B., Methods of Modern Mathematical Physics, I. Functional Analsis, New York London 1972; p. 357.

[2] Il'in V.A., Spectral theory of differential operators. Moscow. «Nauka», 1991, p. 367.

[3] Reed M., Simon B, Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-adjoiness, Academic Press New York London, 1978; p. 394.

[4] Rakhimov A.A., Kamran Zakariya. On an Estimation of Eigenfunctions of Schrödinger's Operatorin a Closed Domain, MathDigest, UPM, Malaysia, June edition, 2010; p. 1-4.

[5] Alimov Sh. A., Il'in V. A., Nikishin E.M., Convergence problems of multiple trigonometric series and spectral decompositions. I, Uspekhi Mat. Nauk, 1976; 31: 28-83.

[6] Rakhimov A. A. On the uniformly convergence spectral expansions of continuous functions in a closed domain. J. Izvestiya of Uzbek Academy of Science, 1987; 6: 17-22.

[7] Alimov Sh. A., Rakhimov A.A. On the localization spectral expansions in a closed domain, J. Differential Equations 1997; 33: 80-2.

[8] Alimov Sh. A., Rakhimov A.A.: On uniformly convergence of spectral expansions in a closed domain, Doclades of Uzbek Acad Science, 1986; 10: 5-7.

[9] Khalmukhamedov A.R. Spectral theory of elliptic equations with singular coefficients, Doctoral Thesis, Tashkent, 1998, p. 1-205.

[10] Moiseev E.I. The uniform convergence of certain expansions in a closed domain, Soviet Math Dokl 1977; 18: 549-53.

Received on 28-12-2011 Accepted on 11-01-2011 Published on 22-10-2012

http://dx.doi.org/10.6000/1927-5129.2012.08.02.56