Reformulation of Adams-Moulton Block Methods as a Sub-Class of Two Step Runge-Kutta Method

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Abstract: Adams-Moulton methods for *k* = 2 and *k* = 3 were constructed together with their continuous forms using multi-step collocation methods. The continuous forms were then evaluated at various grid points to produce the block Adams-Moulton methods.

The block methods were then reformulated as a sub-class of two step Runge-Kutta methods (TSRK). Both the Adams and the reformulated methods were applied to solve initial value problems and the reformulated methods proved superior in terms of stability.

Keywords: Reformulation, Adams-Moulton, block method, two-step, Runge-Kutta and collocation.

INTRODUCTION

The method for the numerical solution of Initial Value Problem (IVP) is as important as the solution itself. There several methods adopted for the numerical solution of IVP viz:Adams- Multon, Runge-Kutta Euler's rule etc. They all have their inherent advantages and disadvantages. The Euler's rule is known explicit, one-step method and being a one-step method it requires no additional starting values. Its low order makes it of limited practical value.

Linear multi-step methods achieve higher order by sacrificing the one-step nature while retaining linearity wrt, $y_{n+j}, f_{n+j}, j = 0,1,\ldots k$. Higher order can be can also be achieved by sacrificing linearity but preserving the one-step nature. This is the philosophy behind the method proposed by Runge and subsequently developed by Kutta and Heun.

Chollom, J.P [1] (Africa Journal online) vol. 2 number 3 (2007) "on some properties of block linear multi-step methods" worked on the properties of block linear multi-step methods and its application to IVPs. Also in 2003, Chollom, J.P and Jackiewicz, Z [2] worked on the construction of two-step Runge-Kutta methods with large region of absolute stability. In their work, they showed that Adams-Moulton block methods can be reformulated as a sub-class of two-step Runge-Kutta method.

In this work I considered block *k* = 2 and *k* = 3 and using the same reformulation technique applied the scheme to the solution of IVP. The block Adams-Moulton methods were constructed using the multi-step collocation.

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1. DERIVATION OF ADAMS-MOULTON

Methods with Continuous Coefficients

Consider the discrete schemes of Adams-Moulton method for *K* = 2 and *K* = 3 given by.

$$y_{n+2} - y_{n+1} = \frac{h}{12}\left[5f_{n+2} + 8f_{n+1} - f_n\right] \qquad 1.1.0$$

$$y_{n+3} - y_{n+2} = \frac{h}{24}\left[9 + 19f_{n+1} - 5f_{n+1} + f_n\right] \qquad 1.1.1$$

The continuous scheme of equations 1.1.0 and 1.1.1. are more desirable from both practical and theoretical considerations. In development of N a linear multistep method with continuous coefficients by the use of multistep collocation (Onumanyi *et al*. 1994) [3] we considered an extended set:

$$Q = \{x_n, x_{n+1}, x_{n+k}\} \cup \{x_{n+k-1}, x_{n+k-2}, \ldots x_{n+k}\}$$

From which to select the collocation points.

The necessary collocation and interpolation conditions were then employed directly by the use of matrix inversion without any involvement of integration process. We let

$$y(x) = \sum_{j=0}^{K-1} \alpha_j(x) y_{n+1} + h\left[\sum_{j=0}^{m-1} \beta_j(x) f(x_{j+1}) y(x_{i+1})\right] \qquad 1.1.2$$

$$\left.\begin{aligned} \alpha j(x) &= \sum_{i=0}^{k+m-1} \alpha_{i,i+1,k} \\ h\beta_i(x) &= \sum_{i=0}^{k+m-1} h\beta_{i,i+1,k} \end{aligned}\right\} \qquad 1.1.3$$

x_{n+j} are assumed polynomials and $r(0 \le r \le k)$ are chosen interpolation points. $(x_n, \ldots\ldots, x_{n+k-1})$ and $(x_i, \ldots\ldots, x_n)$ are collocation points taken from Q.

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To determine the $\alpha_j(x)$ and $\beta_j(x)$ in equations 1.1.2 and 1.1.3, conditions

$$y(x_{n+j}) = y_{n+j}, j \in [0,1\ldots.,k-1] \quad 1.1.4$$

and

$$y'(x_j) = f[x_j, y(x_j)], j = 1,\ldots,m \quad 1.1.5$$

Satisfy the polynomial equation 1.1.3

Form the interpolation conditions of 1.1.4 and 1.1.5 the expression in $y(x)$ in equation 1.1.3 has the following conditions imposed on $\alpha_j(x)$ and $\beta_i(x)$.

$$\alpha_j(x_{n+j}) = \delta_j, \{j = 0,1,\ldots..,k-1, \quad i = 0,\ldots..,k-1\} \quad 1.1.6$$

$$h\beta_i(x_{n+j}) = 0, \{j = 0,1,\ldots..,m-1, \quad i = 0,2\ldots,k-1\} \quad 1.1.7$$

Writing equation 1.1.6 and 1.1.7 in matrix equation form:-

$$DC = I \quad 1.1.8$$

Where *I* is the identity matrix of dimension $K+M$.

Letting

$M = K+1.K > 0$ and $t = 1$ and $x_j = x_{n+j-1}, j = 1,\ldots,K+1$ the left hand side of equation 1.1.8 becomes.

$$DC = \begin{bmatrix} 1 & x_{n+k-1} & x_{n+k-1}^2 & x_{n+k}^{k+1} \\ 0 & 1 & 2x_n & k+1x_k \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+k-1} & (k+1)x_{n+k}^k \end{bmatrix}$$

From equation 1.1.8, the columns of $C = D^{-1}$.

Gives the continuous coefficient of $\alpha_{k-1}(x)$ and $\beta_j(x), j = 0,1\ldots\ldots,k$ and Adam-Moulton scheme of order 1-5 are recovered.

1.2. Reformulation of ADAM-Moulton Scheme as a Subclass of Two –Step Runge-Kutta Method

Recently, Z. Jackewicz and Tracogna [4] introduced a promising class of two-step Runge-Kutta (TSRK) methods for the numerical solution

$$\begin{bmatrix} \alpha_{k-1},1 & h\beta_0 & h\beta_{k,1} \\ (\alpha_{k-1})k-1,2 & h\beta_0,2 & h\beta_{k-1},2 \\ \vdots & \vdots & \vdots \\ (\alpha_{k-1})k+2 & h\beta_0,k+2 & h\beta_k,k+2 \end{bmatrix} \quad 1.1.9$$

Of Ordinary Differential Equation (ODEs)

$$y(x) = f(y(x), \quad x \in (x_0,x)), y(x_0) = y_0 \quad 2.0$$

With the exact solution denoted by $y(x)$ these methods depend on stage values at two consecutive steps and have the form

$$\left.\begin{aligned} y_{i+1} &= \theta y_{i-1} + (1-\theta)y_i + h\sum_{j=1}^{s}\left(vjf(r^j - i - 1) + w_j f(Y_i^j\right) \\ Y_i^j &= u_j y_{i-1} + (1-\theta uj)y_i + h\sum_{j=1}^{s}\left(\alpha_j kf(Y_{i-u}^k)b_j kf(Y_i^k\right) \end{aligned}\right\} \quad 2.1$$

$$J = 1,2,\ldots,S, \quad I = 1,2,\ldots..,N-1$$

where N is an integer greater than zero, $h = \frac{x-x_0}{N}$ is a fixed step size,

$x_i = x_0 + ih, i = 0,1,..N$ is an approximation (possible of lower order) to

$$y(x_i + c_j h)$$

These methods generalized a class of (explicit) pseudo Runge-Kutta formulae.

The equation 2.1 can be represented in a table given below.

U	A	B
θ	V	W

$$= \begin{array}{cccccccc} U_1 & a_{11} & a_{12} & \cdots & a_{1s} & b_{11} & b_{12} & \ldots & b_{1s} \\ U_2 & a_{21} & a_{22} & \cdots & a_{2s} & b_{21} & b_{22} & \ldots & b_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ U_S & a_{s1} & a_{s2} & \cdots & a_{ss} & b_{s1} & b_{s2} & \ldots & b_{ss} \\ \theta & V_1 & V_2 & \ldots & V_s & W_1 & W_2 & \ldots & W_s \end{array}$$

2. DERIVATION OF THE BLOCKS METHODS

The block methods are derived by evaluating the derivative of the continuous schemes obtained from the matrix inversion technique at $x = x_n, x_{n+1},\ldots\ldots,x_{n+6}$. In this research, the cases of *k* = 2 and *k* = 3 are considered. Thus for *k* = 2 the matrix D is given by

$$D = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 \end{bmatrix}$$

Since $DC = I$ in 1.1.8, we use the following formula to obtain the elements of C^{-1} thus.

$$U_{ij} = \frac{d_{ij}}{L_{1i}}; L_{i1} = d_{ij},$$

$L_{ij} = d_{i1} - \Sigma L_{ik}U_{ik}$, etc.

Hence

$$C_{11} = 1, C_{12} = \frac{-2x_{n+1}^3 - 3hx_{n+1}^2}{12h^2}, C_{13} = \frac{x_{n+1}^3 - 3hx_{n+1}^2}{12h^2}, C_{14} = \frac{3hx_{n+1}^3 - 2x_{n+1}^3}{12h^2}$$
$$C_{21} = 0, C_{22} = \frac{x_{n+1}^2 + hx_{n+1}}{2h^2}, C_{23} = \frac{h^2 - x_{n+1}^2}{2h^2}, C_{24} = \frac{x_{n+1}^2 - hx_{n+1}^3}{2h^2},$$
$$C_{31} = 0, C_{32} = \frac{-h - 2x_{n+1}}{4h^2}, C_{33} = \frac{x_{n+1}}{h^2}, C_{34} = \frac{h - 2x_{n+1}}{4h^2},$$
$$C_{41} = 0, C_{42} = \frac{1}{6h^2}, C_{43} = \frac{-1}{3h^{2'}}, C_{44} = \frac{1}{6h^{2'}}$$

To derive the continuous form of Adams –Moulton method for $k = 2$, we represent the general form of the two-step method as

$$y(x) = \alpha_1(x)y_{n+1} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+1}] \quad 3.1$$

where

$$\left.\begin{aligned} \alpha_1(x) &= C_{11} + C_{12}x + C_{31}x^2 + C_{41}x^3 \\ \beta_0(x) &= C_{21} + C_{22}x + C_{32}x^2 + C_{42}2x^3 \\ \beta_1(x) &= C_{31} + C_{23}x + C_{33}x^2 + C_{43}2x^3 \\ \beta_2(x) &= C_{41} + C_{24}x + C_{34}x^2 + C_{44}2x^3 \end{aligned}\right\} \quad 3.2$$

On substituting the elements of C^{-1} into 3.2 yields the following;

$$\alpha_1(x) = 1$$

$$\left.\begin{aligned} \beta_0(x) &= \frac{2(x - x_{n+1})^3 - 3h^2(x - x_{n+1})^2}{12h^2} \\ \beta_1(x) &= \frac{-(x - x_{n+1})^3 - 3h^2(x - x_{n+1})}{3h^2} \\ \beta_2(x) &= \frac{(x - x_{n+1})^3 - 3h(x - x_{n+1})^2}{12h^2} \end{aligned}\right\}^{-} \quad 3.3$$

Substituting the value of 3.3 into 3.1 yields the following as the continuous form of Adams-Moulton method for $k = 2$

$$\left.\begin{aligned} y(x) = yn + 1 &+ \frac{2(x - x_{n+1})^3 - 3h(x - x_{n+1})^2}{12h^2} \\ &+ \left[\frac{-(x - x_{n+1})^3 - 3h^2(x - x_{n+1})}{3h^2}\right]f_{n+1} \\ &+ \frac{2(x - x_{n+1})^3 - 3h(x - x_{n+1})^2}{12h^2} \end{aligned}\right\} \quad 3.4$$

Evaluating 3.4 at $x = x_n$ and $x = x_{n+2}$ we obtained the following two schemes respectively.

$$y_{n+1} - y_n = \frac{h}{12}[-f_{n+2} + 8f_{n+1} + 5f_n]$$

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

For $K = 3$

$$D = \begin{bmatrix} 1 & x_{n+2} & x_{n+1}^2 & x_{n+2}^3 & x_{n+2}^3 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^3 & 4x_{n+2}^3 \end{bmatrix}$$

Also by similar procedure, we obtained the elements of C^{-1} as:

$$C_{11} = 1, \quad C_{12} = \frac{x^2 - 8h^2x_n^2 - 24h^2x_n - 8h^4}{12h^3},$$
$$C_{13} = \frac{3x^4 + 20hx_n^3 + 36h^2x_n^2 - 32h^4}{24h^3},$$
$$C_{14} = \frac{-3x_n^4 - 16hx_n^3 - 18h^2x_n^2 - 8h^4}{12h^3}, C_{15} = \frac{x_n^4 + hx_n^3 + 4h^2x_n^2}{24h^3}$$
$$C_{21} = 0, C_{22} = \frac{x_n^3 + 6hx_n^2 + 11h^2x_n}{6h^3}, C_{23} = \frac{-x_n^3 - 5hx_n^2 - 6h^2x_n^2}{2h^3},$$
$$C_{24} = \frac{x_n^3 + 2hx_n^2 + 3h^2x_n^2}{2h^3}, C_{25} = \frac{-x_n^3 - 3hx_n^2 - 2h^2x_n}{6h^3},$$
$$C_{31} = 0, C_{32} = \frac{-(3x_n^2 + 12hx_n + 11h^2)}{12h^3}, C_{33} = \frac{3x_n^2 + 10hx_n + 6h^2}{12h^3},$$
$$C_{34} = \frac{-(3x_n^2 + 8hx_n + 3h^2)}{4h^3}, C_{35} = \frac{3x_n^2 + 6hx_n + 2h^2}{12h^3},$$
$$C_{41} = 0, C_{42} = \frac{x_n + 2h}{6h^3}, C_{43} = \frac{-(3x_n + 5h)}{6h^3}, C_{44} = \frac{3x_n + 4h}{6h^3},$$
$$C_{45} = \frac{-(x_n + h)}{6h^3},$$
$$C_{51} = 0, C_{52} = \frac{-1}{24h^3}, C_{53} = \frac{1}{8h^3}, C_{55} = \frac{1}{24h^3},$$

Also, the general form for the continuous form is given as:

$$y(x) = \alpha_2(x)y_{n+2} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}] \quad 3.5$$

Also by similar procedure, we have that,

$$\left.\begin{aligned} \beta_0(x) &= \frac{-(x - x_n)^4 + 8h(x - x_n)^3 - 22h^2(x - x_n)^2 + 24h^3(x - x_n) - 8h^4}{24h^3} \\ \beta_1(x) &= \frac{3(x - x_n)^4 - 20h(x - x_n)^3 + 36h^2(x - x_n)^2 - 32h^4}{24h^3} \\ \beta_2(x) &= \frac{-3(x - x_n)^4 + 16h(x - x_n)^3 - 18h^2(x - x_n)^2 - 8h^4}{24h^3} \\ \beta_3(x) &= \frac{(x - x_n)^4 + 4h(x - x_n)^3 - 4h^2(x - x_n)^2}{24h^3} \end{aligned}\right\} \quad 3.6$$

On substituting 3.6 into 3.5 we have the continuous form of the Adams-Moulton method for $K = 3$ as

$$\left.\begin{aligned} y(x) = y_{n+2} &+ [-(x-x_n)^4 + 8h(x-x_n)^3 - 22h^2(x-x_n)^2 + 24h^3(x-x_n) - 8h^4]f_n \\ &+ [3(x-x_n)^4 - 20h(x-x_n)^3 + 36h^2(x-x_n)^2 - 32h^4]f_{n+1} \\ &+ [-3(x-x_n)^4 + 16h(x-x_n)^3 - 18h^2(x-x_n)^2 - 8h^4]f_{n+2} \\ &+ [(x-x_n)^4 + 4h(x-x_n)^3 - 4h^2(x-x_n)^2]f_{n+3} \end{aligned}\right\} \quad 3.7$$

Evaluating 3.7 at $x = x_{n+3}, x_{n+2}$ and x_n, yields the following three discrete schemes.

$$y_{n+3} - y_{n+2} = \frac{h}{24}[9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n]$$

$$y_{n+3} - y_{n+1} = \frac{h}{24}[-f_{n+3} + 13f_{n+2} - 13f_{n+1} - f_n]$$

$$y_{n+3} - y_n = \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n]$$

3. REFORMULATION OF ADAMS-MOULTON METHODS AS A SUBCLASS OF TWO- STEP RUNGE-KUTTA METHOD

The reformulation equation is given in 2.1, thus we define

$$y_{n+3} = Y_3^{(n)}, y_{n+2} = Y_2^{(n)}, y_{n+1} = Y_1^{(n)}, \text{and } y_n = Y^{(n-1)},$$

For $k = 2$, we have $y_{n+1} - y_n = \frac{h}{12}[-f_{n+2} + 8f_{n+1} + 5f_n]$

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

Reformulating, $Y_1^{(n)} = y_n + \frac{h}{12}\left[-f(Y_2^{(n)}) + f(Y_1^{(n)}) + f(Y^{(n-1)})\right]$

$$Y_2^{(n)} = y_{n+1} + \frac{h}{12}\left[5f(Y_2^{(n)}) + 8f(Y_1^{(n)}) - f(Y^{(n-1)})\right]$$

$$y_{n+2} = y_{n+1} + \frac{h}{12}\left[5f(Y_2^{(n)}) + 8f(Y_1^{(n)}) - f(Y^{(n-1)})\right]$$

1	$\frac{5}{12}$	0	$\frac{8}{12}$ $\frac{-1}{12}$
1	$\frac{-1}{12}$	0	$\frac{8}{12}$ $\frac{5}{12}$
1	$\frac{-1}{12}$	0	$\frac{8}{12}$ $\frac{-1}{12}$ $\frac{8}{12}$ $\frac{5}{12}$

For $k = 3$, we have

$$y_{n+3} - y_{n+2} = \frac{h}{24}\left[9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n\right]$$

$$y_{n+3} - y_{n+1} = \frac{h}{24}\left[-f_{n+3} + 13f_{n+2} - 13f_{n+1} - f_n\right]$$

$$y_{n+3} - y_n = \frac{h}{3}\left[f_{n+2} + 4f_{n+1} + f_n\right]$$

Reformulating, $Y_3^{(n)} = y_n + \frac{h}{3}\left[f(Y_2^{(n)}) + 4f(Y_1^{(n)}) + f(Y^{(n-1)})\right]$

$$Y_3^{(n)} = y_{n+1} + \frac{h}{24}\left[9f(Y_3^{(n)}) + 13f(Y_2^{(n)}) - 13f(Y_1^{(n)}) - f(Y^{(n-1)})\right]$$

$$Y_3^{(n)} = y_{n+2} + \frac{h}{24}\left[9f(Y_3^{(n)}) + 19f(Y_2^{(n)}) - 5f(Y_1^{(n)}) - f(Y^{(n-1)})\right]$$

$$y_{n+3} = y_{n+2} + \frac{h}{24}\left[9f(Y_3^{(n)}) + 19f(Y_2^{(n)}) - 5f(Y_1^{(n)}) - f(Y^{(n-1)})\right]$$

Its table is given below.

Table 1:

1 0 1	$\frac{1}{3}$ 0 0 $\frac{-1}{24}$ 0 0 $\frac{1}{24}$ 0 0	$\frac{4}{3}$ $\frac{1}{3}$ 0 $\frac{-13}{24}$ $\frac{13}{24}$ $\frac{1}{24}$ $\frac{-5}{24}$ $\frac{19}{24}$ $\frac{9}{24}$
1	$\frac{1}{24}$ 0 0	$\frac{-5}{24}$ $\frac{19}{24}$ $\frac{9}{24}$

4. CONVERGENT ANALYSIS

The necessary and sufficient condition for a linear multistep method (LMM) to be convergent are that it be zero stable and consistent. (Fundamental theorem of Daliquist on LMM) [5].

Convergent analysis was carried out on both $K = 2$ and $K = 3$ and were found to be convergent and zero-stable. Their order and error constants are given in the table below

Table 2:

Evaluation Point	Order	Error Constant
$y(x_{n+2})$	3	-0.041666667
$y(x_n)$	3	-0.041666667

Their regions of absolute stability are

Table 3:

$y(x_{n+2})$	(-6,0)
$y(x_n)$	(0,6)

Table 4:

Evaluation Point	Order	Error Constant
$y(x_{n+3})$	4	-0.026388889
$y(x_{n+1})$	4	-0.015277778
$y(x_n)$	4	-0.001111111

The region of absolute stability for the above is given below

Table 5:

$y(x_{n+3})$	(-3,0)
$y(x_{n+1})$	(-3,0)
$y(x_n)$	(-3,0)

For the reformulated schemes, their stability regions are: $K = 2$, (0,8) and $K = 2$, (0,4.5).

5. NUMERICAL EXAMPLES

The block Adams-Moulton methods and the reformulated two-step Runge-Kutta method were used to solve the following initial value problems (IVP) considering various step sizes.

$$\left.\begin{array}{c} y' = -60y + 10x \\ y_0 = \frac{1}{6}, \quad y(x) = \frac{1}{6}[x + e^{-60x}] \end{array}\right\} \qquad 5.0.1$$

$$\left.\begin{array}{c} y' = 1 + x - 2y \\ y_0 = 2, \quad y(x) = \frac{1}{4}[2x + 7e^{-2x} + 1] \end{array}\right\} \qquad 5.0.2$$

Now solving 5.0.1 using the block $K = 2$

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

$$y_{n+2} - y_n = \frac{h}{12}[-f_{n+2} + 8f_{n+1} + 5f_n]$$

For $h = 0.05$

Solving the following values were obtained.

$$\begin{aligned} y_1 &= 0.00416666666, \quad y_2 = 0.016666666, \\ y_3 &= 0.045535714, \quad y_4 = 0.0630952338, \\ y_5 &= 0.077699829, \quad y_6 = 0.095493197, \\ y_7 &= 0.111116689, \quad y_8 = 0.128893474, \\ y_9 &= 0.144444035, \quad y_{10} = 0.162221894, \end{aligned}$$

Now solving the same problem using a step size of 0.1, we have.

$$\begin{aligned} y_1 &= -0.032456140, \quad y_2 = 0.075438596, \\ y_3 &= 0.031902123, \quad y_4 = 0.045337027 \\ y_5 &= 0.080174466, \quad y_6 = 0.037755747, \\ y_7 &= 0.122520417, \quad y_8 = 0.03847416, \\ y_9 &= 0.1626829611, \quad y_{10} = 0.042243855, \end{aligned}$$

Using the block for $K = 3$, $h = 0.05$, the following values were obtained.

$$\begin{aligned} y_1 &= -0.004131911, \quad y_2 = 0.041361789 \\ y_3 &= 0.023526423, \quad y_4 = 0.055498339, \\ y_5 &= 0.089003321, \quad y_6 = 0.123035024, \\ y_7 &= 0.102639390, \quad y_8 = 0.144721221 \\ y_9 &= 0.129307559 \quad y_{10} = 0.151114450 \end{aligned}$$

for $h = 0.1$, the results are

$$\begin{aligned} y_1 &= 0.015807560, \quad y_2 = 0.053264605, \\ y_3 &= 0.003436426, \quad y_4 = 0.071562688, \\ y_5 &= 0.074687356, \quad y_6 = 0.110312807, \\ y_7 &= 0.111594664, \quad y_8 = 0.132309964, \\ y_9 &= 0.143308542, \quad y_{10} = 0.16457492. \end{aligned}$$

Table 6: $h = 0.1$

x	*Exact Solution*	$K = 2$	$K = 3$
0.1	0.017079792	−0.032456140	0.015807560
0.2	0.033334357	0.075438596	0.015807560
0.3	0.050000002	0.031902123	0.003436326
0.4	0.066666667	0.045337027	0.071562688
0.5	0.083333333	0.080174466	0.074687356
0.6	0.100000000	0.037755747	0.110312807
0.7	0.116666667	0.1225201416	0.111594664
0.8	0.133333333	0.038471416	0.132309964
0.9	0.150000000	0.162682961	0.143308542
1.0	0.166666667	0.042243855	0.164574792

Table 7: Error Table, *h* = 0.1

x	Error K = 2	Error K = 3
0.1	4.9535932×10^{-2}	1.272232×10^{-3}
0.2	$-4.2104239 \times 10^{-2}$	$-1.9930248 \times 10^{-2}$
0.3	$1.8.97879 \times 10^{-2}$	4.6563576×10^{-2}
0.4	2.1329639×10^{-2}	-4.896021×10^{-3}
0.5	3.158867×10^{-3}	8.645977×10^{-3}
0.6	6.2244253×10^{-2}	$-1.0312807 \times 10^{-2}$
0.7	-5.853751×10^{-3}	5.072003×10^{-3}
0.8	9.4861917×10^{-2}	1.023369×10^{-3}
0.9	$-1.2682961 \times 10^{-2}$	6.691458×10^{-3}
1.0	$1.244228115 \times 10^{-1}$	2.091875×10^{-3}

Table 8: *h* = 0.05

x	Exact Solution	K=2	K=3
0.1	0.017079792	0.004166667	0.004131911
0.2	0.033334357	0.016666667	0.041361789
0.3	0.050000002	0.045535714	0.023526423
0.4	0.066666667	0.063095238	0.055498339
0.5	0.083333333	0.077699829	0.089003321
0.6	0.100000000	0.095493197	0.123035024
0.7	0.116666667	0.11116689	0.102639390
0.8	0.133333333	0.128893474	0.144721221
0.9	0.150000000	0.144444035	0.129307559
1.0	0.166666667	0.162221894	0.151114450

Table 9: Error Table, *h* = 0.1

	x	Error K = 2	Error K = 3
K = 2	0.1	1.2913125×10^{-2}	1.2947881×10^{-2}
	0.2	1.666769×10^{-2}	-8.027432×10^{-3}
	0.3	4.464288×10^{-3}	2.647358×10^{-2}
	0.4	3.571429×10^{-3}	1.1168328×10^{-2}
	0.5	5.633504×10^{-3}	-5.669988×10^{-3}
	0.6	4.506803×10^{-3}	$-2.3035024 \times 10^{-2}$
	0.7	5.549978×10^{-3}	1.4027277×10^{-2}
	0.8	4.439859×10^{-3}	$-1.1387888 \times 10^{-2}$
	0.9	5.555965×10^{-3}	2.0692441×10^{-2}
	1.0	4.444773×10^{-3}	1.5552217×10^{-2}

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

$$y_{n+2} - y_n = \frac{h}{12}[-f_{n+2} + 8f_{n+1} + 5f_n]$$

we have for $h = 0.1$

$$12y_{n+2} - 12y_{n+1}[5(1+x-2y) + 8(1+x-2y) + (1+1x-2y)]$$

Hence

$$13y_2 - 10y_1 = 1.78 \qquad 5.0.10$$

similarly

$$-0.2y_2 + 13y_1 = 23.26 \qquad 5.0.11$$

Solving equations 5.0.10 and 5.0.11

Simultaneously we have

$$y_1 = 1.732692308, \quad y_2 = 1.523076923,$$
$$y_3 = 1.360376162, \quad y_4 = 1.236348267,$$
$$y_5 = 1.143768636, \quad y_6 = 1.07711256,$$
$$y_7 = 1.031537206, \quad y_8 = 1.003339189,$$
$$y_9 = 0.989272193, \quad y_{10} = 0.986853742.$$

K = 3, *h* = 0.1

$$y_{n+3} - y_{n+2} = \frac{h}{24}[9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n]$$

$$y_{n+3} - y_{n+1} = \frac{h}{24}[-f_{n+3} + 13f_{n+2} - 13f_{n+1} - f_n]$$

$$y_{n+3} - y_n = \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n]$$

Solving we have matrix as

$$\begin{bmatrix} -1 & -20.2 & 25.8 \\ -21 & 26.6 & -0.2 \\ 0.8 & 3.2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 3.16 \\ 6.26 \end{bmatrix}$$

Evaluating the matrix, we have

$$y_1 = 1.732768925, \quad y_2 = 1.523076923,$$
$$y_3 = 1.360408367, \quad y_4 = 1.236304537,$$
$$y_5 = 1.143774937, \quad y_6 = 1.077072736,$$
$$y_7 = 1.031527678, \quad y_8 = 1.003306725,$$
$$y_9 = 0.9892600378, \quad y_{10} = 0.9868244531.$$

Now solving equation 5.0.2 using the reformulated scheme, the values of the free parameters $a_{11}, a_{12}, \ldots, a_{1s}$ and $b_{11}, b_{12}, \ldots, b_{1s}$ are substituted into the continuous TSR-K method as thus, for S=2.

$$y_{i+1} = \theta y_{i+1} + (1-\theta)y_i + h[V_1 f(Y_{1i-1}) + W_1 f(Y_{1i}) + V_2 f(Y_{2i-1}) + W_2 f(Y_{i2})]$$

Table 10: *h* = 0.1

x	Exact Solution	*K* = 2	*K* = 3
0.1	1.732778818	1.732692308	1.732768925
0.2	1.523060081	1.523076923	1.523057769
0.3	1.360420363	1.360376162	1.360408367
0.4	1.236325687	1.236348267	1.236304537
0.5	1.143789022	1.143768636	1.143774937
0.6	1.077089871	1.077112560	1.077072736
0.7	1.031544687	1.031537206	1.031527678
0.8	1.003318906	1.003339189	1.003306725
0.9	0.989273054	0.989272193	0.989260038
1.0	0.986836746	0.986853742	0.986824453

Table 11: Error *h* = 0.1

X	K = 2	K = 3
0.1	8.651×10^{-5}	9.893×10^{-6}
0.2	-1.6842×10^{-5}	2.312×10^{-6}
0.3	4.4201×10^{-5}	1.1996×10^{-5}
0.4	-2.258×10^{-5}	2.115×10^{-5}
0.5	2.0386×10^{-5}	1.4085×10^{-5}
0.6	-2.2689×10^{-5}	1.7135×10^{-5}
0.7	7.481×10^{-6}	1.7009×10^{-5}
0.8	-2.0283×10^{-5}	1.2181×10^{-5}
0.9	8.61×10^{-7}	1.3016×10^{-5}
1.0	-1.6996×10^{-5}	1.22929×10^{-5}

$$y_i^{(1)} = U_1 y_{i-1} + (1-U_1) y_i + h[a_{11} f(Y_{1i-1}) + b_{11} f(Y_{1i}) + a_{12} f(Y_{2i-1}) + b_{12} f(Y_{i2})]$$

$$y_i^{2} = U_2 y_{i-1} + (1-U_{12}) y_i + h[a_{21} f(Y_{1i-1}) + a_{22} f(Y_{2i-1}) + a_{21} f(Y_{i1}) + b_{22} f(Y_{i2})]$$

S = 3

$$y_{i+1} = \theta y_{i-1} + (1-\theta) y_i + h[V_1 f(Y_{1i-1}) + W_1 f(Y_{1i}) + V_2 f(Y_{2i-1}) + W_2 f(Y_{i2}) + V_3 f(Y_{i3})]$$

$$y_i^{(1)} = U_1 y_{i-1} + (1-U_1) y_i + h[a_{11} f(Y_{i-1}) + b_{12} f(Y_{2i-1}) + a_{23} f(Y_{3(i-1)}) + b_{11} f(Y_{i-1}) + b]$$

$$y_i^{(2)} = U_1 y_{i-1} + (1-U_2) y_1 + h[a_{21} f(Y_{1i-1}) + a_{22} f(Y_{2i-1}) + b_{23} f(Y_{i-1}) + b_{21} f(Y_{i-1}) + b]$$

$$y_i^{(3)} = U_3 y_{i-1} + (1-U_3) y_i + h[a_{31} f(Y_{i-1}) + a_{32} f(Y_{2(i-1)}) + a_{33} f(Y_{3(i-1)}) + b_{31} f(Y_{i-1}) + b]$$

Thus solving 5.0.2, the following results were obtained as contained in Table **7**.

Table 12: {h = 0.1}, Reformulated Scheme

X	Exact Solution	*K* = 2	*K* = 3
0	2.000000000	2.000000000	2.000000000
0.1	1.732778818	1.700000000	1.700000000
0.2	1.523960081	1.451748148	1.641084610
0.3	1.360420363	1.308771208	1.448463233
0.4	1.236325687	1.203082244	1.304033029
0.5	1.143789022	1.281327248	1.192758695
0.6	1.077089871	1.077321208	1.112191440
0.7	1.031544687	1.045801716	1.056080923
0.8	1.003318906	1.029687970	1.019872305
0.9	0.989273054	1.025879662	0.999759807
1.0	0.986836745	1.031900258	0.992681275

Table 13: Error Reformulated Scheme (*h* = 0.1)

X	K = 2	K = 3
0	0.000000000	0.000000000
0.1	3.2778818×10^{-2}	3.2778818×10^{-2}
0.2	7.2211933×10^{-2}	$-1.17124529 \times 10^{-1}$
0.3	5.1649155×10^{-2}	-8.804287×10^{-2}
0.4	3.3243443×10^{-2}	$-6.7707342 \times 10^{-2}$
0.5	$-1.37538226 \times 10^{-1}$	$-4.8969673 \times 10^{-2}$
0.6	-2.31337×10^{-4}	$-3.5101569 \times 10^{-2}$
0.7	$-1.4257029 \times 10^{-2}$	$-2.4536236 \times 10^{-2}$
0.8	$-2.6369064 \times 10^{-2}$	$-1.6553399 \times 10^{-2}$
0.9	$-3.6606608 \times 10^{-2}$	$-1.0486753 \times 10^{-2}$
1.0	$-4.5063513 \times 10^{-2}$	-5.84453×10^{-3}

I made use of the block two-step and three-step Adam Moulton methods reformulated as a subclass of two-step Runge-Kutta method. The block methods were derived by evaluating Adam-Moulton continuous schemes at $x = x_n, x_{n+1}$ and x_{n+3}.

The methods are of degree K and order k+1 and their blocks are also stable. However, the reformulated methods have gained stability because for *k* = 2 it is A-stable and *k* = 3 is now A_∞ - stable.

The block for $k = 3$ shows it is a better scheme than the $k = 2$ as demonstrated by their errors in the solved numerical examples. The block $k = 3$ also yielded an equivalent Simpson's scheme which is the most accurate linear multistep method (LMM) i.e. $y_n = y_{n+2} + \frac{h}{3}\{f_{n+2} + 4f_{n+1} + f_n\}$.

CONCLUSION

The block methods were shown to have lower errors than their conventional schemes and exhibited similar characteristic convergent properties to their respective discrete schemes usually applied singularly.

The advantage of this approach is that it reduces computational efforts and speeds up computation. Also by the gain in stability, the reformulated schemes are better numerical methods.

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