A Class of Flows for Couple Stress Fluids

Rana Khalid Naeem*

Department of Mathematics, University of Karachi, Karachi, Pakistan

Abstract: Exact solutions are derived for a class of two dimensional couple stress flows. This class consists of flows for which the vorticity distribution is characterized by the Eq. (12). The solutions are obtained by introducing the functions Ψ, H and the canonical transformation. The effects of the parameters K, m, U, A, B, D, E, L, and M on velocity components u and v are discussed and streamlines for the various values of the parameters are also presented.

Keywords: Couple stress fluids; Exact solutions; Effect of parameters on fluid flow.

1. INTRODUCTION

The accurate flow behavior of couple stress fluids cannot be predicted using classical Newtonian theory and therefore models are developed for these fluids. However, the model developed by Stokes (1966) is widely used because of its mathematical simplicity [1]. The study of the couple stress fluids is of great interest due to their applications in science, engineering and industries. The readers interested in the work on couple stress fluids and their applications may refer to [2-13] and the references therein.

The objective of this paper is to derive a class of two dimensional couple stress flows and discuss the effect of the pertinent parameters m, K, U, A, B, D, E, L and M on velocity components u and v and present streamlines for flows for different values of the parameters. This class consists of flows for which the velocity distribution is characterized by the Eq. (12). To achieve our objective, the basic flow equations are expressed interms of the function Ψ, the function H and the canonical transformation defined in section 3. The rest of the paper is divided into five sections. In section 2, the basic flow equations are presented. In section 3, the transformed flow equations are presented and their solutions are determined. In section 4 we sum up the work up to section 3. In section 5, the effect of the parameters of interest on the velocity components u and v are discussed and the streamlines for various values of these parameters are also presented. Conclusions are given in section 6.

2. FLOW EQUATIONS

The equations governing the motion of incompressible couple stress fluids in the absence of body forces are

$$\nabla . u = 0 \tag{1}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}.\nabla)\boldsymbol{u}\right)=\nabla p+\mu\nabla^2\boldsymbol{u}-\eta\nabla^2\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{u}$ is the velocity vector, ρ is the constant density, p is the pressure, μ is the coefficient of viscosity and η the material constant responsible for the couples stress parameter [1].

For flows in xy-plane Eqs. (1) and (2) become

$$u_x + v_y = 0, \tag{3}$$

$$h_x - \rho v\omega = -\mu\omega_y + \eta(\nabla^2\omega)_y, \tag{4}$$

$$h_y + \rho u\omega = \mu\omega_x - \eta(\nabla^2\omega)_x, \tag{5}$$

In Eqs. (4) and (5), h is the generalized pressure and ω is the vorticity function. These functions are defined as

$$h = \frac{\rho}{2}(u^2+v^2)+p, \tag{6}$$

$$\omega = v_x - u_y. \tag{7}$$

Equation (3) implies the existence of the stream function ψ such that

$$u = \psi_y, \qquad v = -\psi_x, \tag{8}$$

Inserting Eq. (8) in Eqs. (4-6), we get

$$h_x + \rho\psi_x\omega = -\mu\omega_y + \eta(\nabla^2\omega)_y, \tag{9}$$

$$h_y + \rho\psi_y\omega = \mu\omega_x - \eta(\nabla^2\omega)_x, \tag{10}$$

$$\omega = -\nabla^2\psi, \tag{11}$$

3. SOLUTIONS

In this section, we determine a class of flows for which the vorticity distribution is characterized by the equation

*Address corresponding to this author at the Department of Mathematics, University of Karachi, Karachi, Pakistan; E-mail: 2rknaeem@gmail.com

ISSN: 1814-8085 / E-ISSN: 1927-5129/12

$$\nabla^2\psi = K(\psi - Ux - Uy), \tag{12}$$

where K and U are both non zero constants.

By setting $\Psi = \psi - U_x - U_y$ Eq. (12) becomes

$$\nabla^2\Psi = K\psi. \tag{13}$$

Eqs. (9-11), utilizing Eqs. (12) and (13), become

$$H_x = \rho U(K\Psi + m\Psi_y), \tag{14}$$

$$H_y = \rho U(K\Psi - m\Psi_x), \tag{15}$$

$$\omega = -K\Psi \tag{16}$$

In Eqs. (14) and (15), the function H and m are

$$H = h - \frac{\rho K\Psi}{2}, \tag{17}$$

$$m = \frac{\mu K - \eta K^2}{\rho U} \tag{18}$$

On introducing the canonical coordinates

$$\xi = x + y, \qquad \eta = y, \tag{19}$$

the Eqs. (13-15) become

$$2\Psi_{\xi\xi} + 2\Psi_{\xi\eta} + \Psi_{\eta\eta} = K\Psi, \tag{20}$$

$$H_\xi = \rho U(K\Psi + m\Psi_\xi + m\Psi_\eta), \tag{21}$$

$$H_\eta = -\rho Um(2\Psi_\xi + \Psi_\eta). \tag{22}$$

Let us now determine the solutions of Eqs. (20-22).

Eliminating the function H from Eqs. (21) and (22) employing the integrability condition $H_{\xi\eta} = H_{\eta\xi}$,, we get

$$\Psi_\eta + m\Psi = 0, \tag{23}$$

whose solution is

$$\Psi = g(\xi)e^{-m\eta}, \tag{24}$$

where $g(\xi)$ is to be determined. Inserting Eq. (24) in Eq. (20), we obtain

$$2g''(\xi) - 2mg'(\xi) + (m^2 - K)g = 0. \tag{25}$$

The solutions of Eq. (25) are

$$\text{I. } g(\xi) = -\frac{1}{m}A + Be^{m\xi} \text{ when } K = m^2. \tag{26}$$

$$\text{II. } g(\xi) = De^{m_1\zeta} + Ee^{m_2\xi}, \text{ when } 2K > m^2. \tag{27}$$

$$\text{III. } g(\xi) = L\cos\frac{\lambda}{2}\xi + M\sin\frac{\lambda}{2}\xi, \quad \text{when } 2K - m^2 < 0, 2K - m^2 = -\lambda^2, \quad \lambda > 0 \tag{28}$$

$$\text{IV. } g(\xi) = (P + Q\xi)e^{\frac{m}{2}\xi}, \text{ when } K = \frac{m^2}{2}. \tag{29}$$

where

$$m_{1,2} = \frac{m \pm \sqrt{2K - m^2}}{2} \tag{30}$$

where $A, B, D, E, L, M, N, P,$ and Q are constants.

Case I: For $K = m^2$, Eq. (24) gives

$$\Psi = -\frac{1}{m}Ae^{-m\eta} + Be^{m(\xi-\eta)}. \tag{31}$$

On inserting Eq. (31) in Eqs. (21) and (22), we get

$$H_\xi = \rho UKBe^{m(\xi-\eta)}, \tag{32}$$

$$H_\eta = -\rho UBm^2e^{m(\xi-\eta)} - \rho UmAe^{-m\eta}, \tag{33}$$

The solution of Eqs. (32) and (33) is

$$H = \frac{\rho UK}{m}Be^{m(\xi-\eta)} + \rho UAe^{-m\eta} + C \tag{34}$$

Equation (18), on substituting $K = m^2$, gives,

$$U = v_1m - v_2m^3 \tag{35}$$

The streamfunction ψ, velocity components and pressure in this case are

$$\psi = (v_1m - v_2m^3)(x + y) - \frac{1}{m}Ae^{-my} + Be^{mx}, \tag{36}$$

$$u = v_1m - v_2m^3 + Ae^{-my}, \tag{37}$$

$$v = -v_1m + v_2m^3 - mBe^{mx}, \tag{38}$$

$$p = -\frac{\rho}{2}(u^2 + v^2) + \frac{\rho K(v_1m - v_2m^3)}{m}Be^{mx} + \rho A(v_1m - v_2m^3)e^{-my} + \frac{\rho K}{2}\left(Be^{mx} - \frac{1}{m}Ae^{-my}\right) + C \tag{39}$$

Case II: For $2K > m^2$, Eq. (24) provides

$$\Psi = De^{m_1\zeta - m\eta} + Ee^{m_2\zeta - m\eta}. \tag{40}$$

Equations (21) and (22), employing Eq. (40), becomes

$$H_\xi = \rho UD(K + mm_1 - m^2)e^{m_1\xi - m\eta} + \rho UE(K + mm_2 - m^2)e^{m_2\xi - m\eta}, \tag{41}$$

$$H_\eta = -\rho UmD(2m_1 - m)e^{m_1\xi - m\eta} - \rho UmE(2m_2 - m)e^{m_2\xi - m\eta}, \tag{42}$$

The solution of Eqs. (41) and (42) is

$$H = \rho UD(2m_1 - m)e^{m_1\xi - m\eta} + \rho UE(2m_2 - m)e^{m_2\zeta - m\eta} + F. \tag{43}$$

For this case streamfunction ψ, velocity components and pressure are

$$\psi = U(x+y) + De^{\frac{m+\sqrt{2K-m^2}}{2}x + \frac{-m+\sqrt{2K-m^2}}{2}y} + Ee^{\frac{m-\sqrt{2K-m^2}}{2}x + \frac{-m-\sqrt{2K-m^2}}{2}y}, \tag{44}$$

$$u = U + \frac{-m+\sqrt{2K-m^2}}{2}De^{\frac{m+\sqrt{2K-m^2}}{2}x + \frac{-m+\sqrt{2K-m^2}}{2}y} + \frac{-m-\sqrt{2K-m^2}}{2}Ee^{\frac{m-\sqrt{2K-m^2}}{2}x + \frac{-m-\sqrt{2K-m^2}}{2}y}, \tag{45}$$

$$v = -U - \frac{m+\sqrt{2K-m^2}}{2}De^{\frac{m+\sqrt{2K-m^2}}{2}x + \frac{-m+\sqrt{2K-m^2}}{2}y} - \frac{m-\sqrt{2K-m^2}}{2}Ee^{\frac{m-\sqrt{2K-m^2}}{2}x + \frac{-m-\sqrt{2K-m^2}}{2}y}, \tag{46}$$

$$p = -\frac{\rho}{2}(u^2+v^2) + \lambda_1 De\frac{m+\sqrt{2K-m^2}}{2}x + \frac{-m+\sqrt{2K-m^2}}{2}y + \lambda_2 Ee\frac{m-\sqrt{2K-m^2}}{2}x + \frac{-m-\sqrt{2K-m^2}}{2}y + F \tag{47}$$

where $\lambda_1 = \frac{\rho K}{2} + \rho U\sqrt{2K-m^2}$ and $\lambda_2 = \frac{\rho K}{2} - \rho U\sqrt{2K-m^2}$.

Case III: When $2K - m^2 < 0$, the streamfunction is

$$\psi = U(x+y) + e^{\frac{m(x-y)}{2}} L\cos\frac{\lambda}{2}(x+y)$$

$$\psi = U(x-y) + e^{\frac{m(x-y)}{2}} L\cos\frac{\lambda}{2}(x+y) + e^{\frac{m(x-y)}{2}} M\sin\frac{\lambda}{2}(x+y) \tag{48}$$

and the exact solution associated to Eq. (48) is

$$u = U - \frac{m}{2}e^{\frac{m(x-y)}{2}} L\cos\frac{\lambda}{2}(x+y) - \frac{m}{2}e^{\frac{m(x-y)}{2}} M\sin\frac{\lambda}{2}(x+y) - e^{\frac{m(x-y)}{2}}\frac{\lambda}{2}L\sin\frac{\lambda}{2}(x+y) + e^{\frac{m(x-y)}{2}}\frac{\lambda}{2}M\cos\frac{\lambda}{2}(x+y), \tag{49}$$

$$v = -U - \frac{m}{2}e^{\frac{m(x-y)}{2}} L\cos\frac{\lambda}{2}(x+y) - \frac{m}{2}e^{\frac{m(x-y)}{2}} M\sin\frac{\lambda}{2}(x+y) + e^{\frac{m(x-y)}{2}}\frac{\lambda}{2}L\sin\frac{\lambda}{2}(x+y) - e^{\frac{m(x-y)}{2}}\frac{\lambda}{2}M\cos\frac{\lambda}{2}(x+y), \tag{50}$$

$$p = -\frac{\rho}{2}(u^2+v^2) + e^{\frac{m(x-y)}{2}}\left(\frac{\rho}{2}L + \rho U\lambda M\right)\cos\frac{\lambda}{2}(x+y) + e^{\frac{m(x-y)}{2}}\left(\frac{\rho}{2}M + \rho U\lambda L\right)\sin\frac{\lambda}{2}(x+y) + N, \tag{51}$$

Case IV: In this case the streamfunction ψ, velocities components and pressure are

$$\psi = U(x+y) + [P + Q(x+y)]\, e^{\frac{m(x-y)}{2}}, \tag{52}$$

$$u = U + Qe^{\frac{m(x-y)}{2}} - \frac{m}{2}[P + Q(x+y)]e^{\frac{m(x-y)}{2}}, \tag{53}$$

$$v = -U - Qe^{\frac{m(x-y)}{2}} - \frac{m}{2}[P + Q(x+y)]e^{\frac{m(x-y)}{2}}, \tag{54}$$

$$p = -\frac{\rho}{2}(u^2 + v^2) + 2\rho Q e^{\frac{m(x-y)}{2}} + \frac{\rho K}{2}[P + Q(x+y)]e^{\frac{m(x-y)}{2}} + R \tag{55}$$

where

$$U = \frac{2v_1 m - v_2 m^3}{4}.$$

4. RESULTS

A class of exact solutions of the equations governing the steady plane flows of incompressible couple stress fluids are determined for which the vorticity distribution is defined by Eq. (12).

5. DISCUSSIONS

The effect of the parameters *K, m, U, A, B, D, E, L,* and *M* on velocity components u and v are depicted in Figures **1-4**, 8-15 and 19-25. The streamlines for the stream function Eqs. (36), (44) and (48) are presented for various values of the parameters.

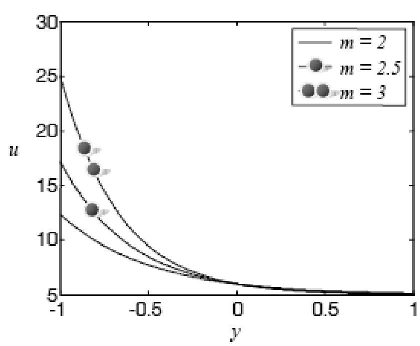

Figure 1: The effect of *m* on velocity component *u* for *U* = 5 and *A* = 1.

Figures **1-4** are for case I. Figures 1 and 2 present the effect of parameters *m* and *A* on the velocity

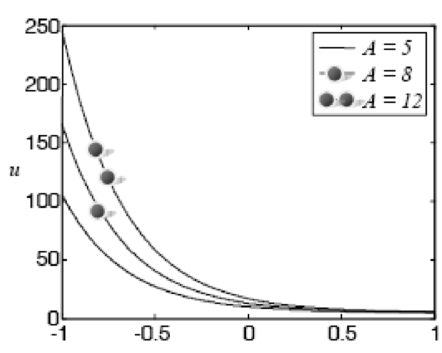

Figure 2: The effect of *A* on velocity component *u* for *U* = 5 and *m* = 1.

component *u*. These figures indicate that the velocity component *u* increases with increase in parameters *m*

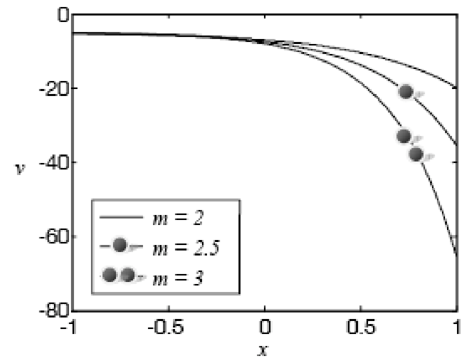

Figure 3: The effect of *m* on velocity component *v* for *U* = 5 and *B* = 1.

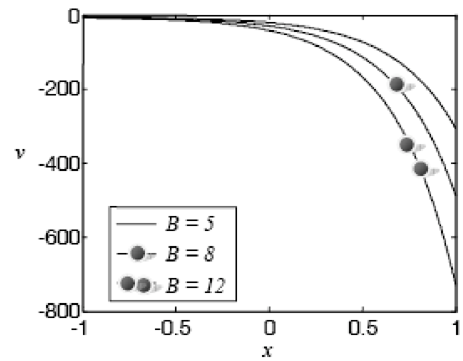

Figure 4: The effect of *B* on velocity component *v* for *U* = 5 and *m* = 3.

and *A*, and increase is larger with increase in *A* than increase in *m*. Figures **3** and **4** illustrate the effect of parameters *m* and *B* on the velocity component *v*. These figures indicate that velocity component *v* increases with increase in *m* and *B* in absolute value. Figures **5-7** represent streamlines for the stream function for case I given by the Eq. (36) for the different values of parameters *m*, *A* and *B*. These figures

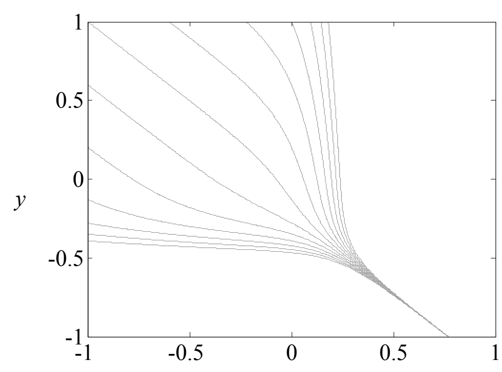

Figure 5: Streamlines for the streamfunction equation (36) for *A* = *B* = 1, *m* = 10, *U* = 5.

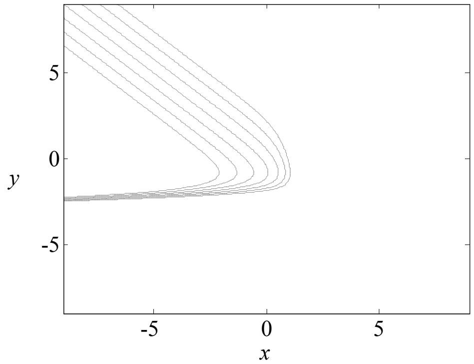

Figure 6: Streamlines for the streamfunction equation (36) for A = 1, B = 1, m = 2, U = 5.

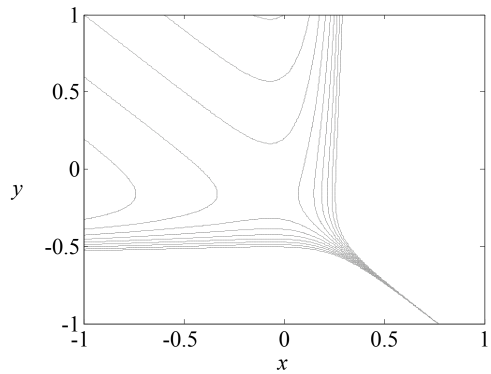

Figure 7: Streamlines for the streamfunction equation (36) for A = B = -1, m = 10, U = 5.

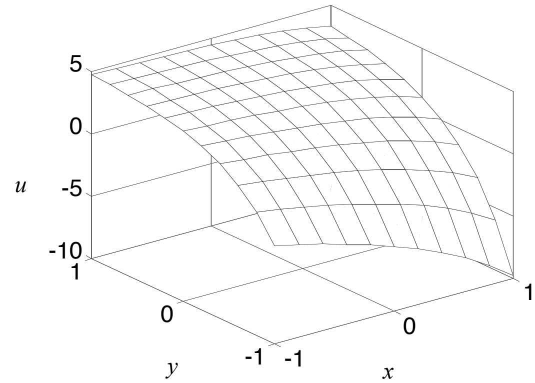

Figure 8: Variation in velocity component u in the flow field for U = 5, K = 3, m = 2, D = E = 1.

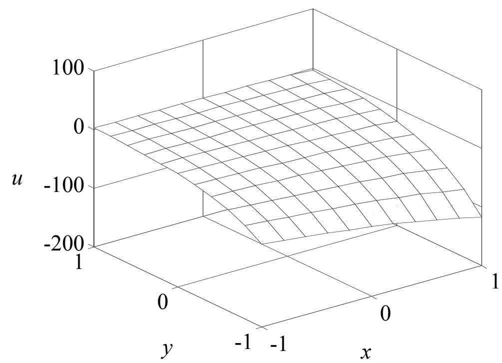

Figure 9: Variation in velocity component u in the field for U = 5, K = 3, m = 3, D = 5, E = 9.

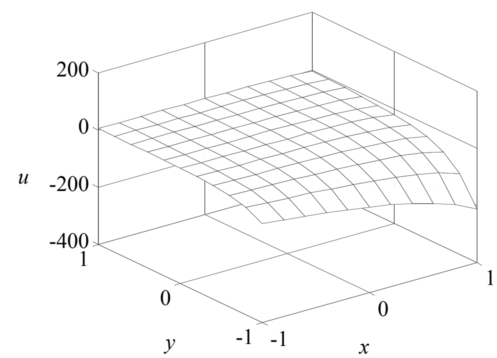

Figure 10: Variation in velocity component u in the flow field for U = 5, K = 12, m = 4, D = E = 1.

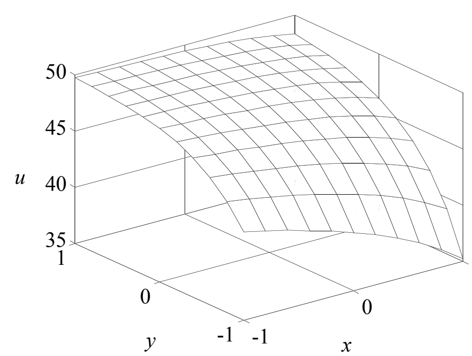

Figure 11: Variation in velocity component u in the flow field for U = 50, K = 3, m = 2, D = E = 1.

indicate that streamline pattern change with change in parameters. We mention that there exists a stagnation point in the flow region for U < 0, A > 0, B > 0 or U > 0, A < 0, B < 0. The stagnation point is

$$(x, y) = \left(\frac{1}{m} \ln \frac{-U}{mB}, \quad -\frac{1}{m} \ln \frac{-U}{A} \right)$$

The figures 8-15 depict the effect of the pertinent parameters m, K, U, D and E on the velocity components u and v for case II. Figures **8** and **9** represent the variation of velocity component u in the flow field for different values of the parameters D and E and fixed values of the other parameters. Comparison of these figures indicates that velocity component u increases with increase in parameters D and E in absolute value. Comparing Figures **8** and **10**, we find that velocity component u increases with increase in parameter m and K. Comparison of Figures **8** and **11** indicate that velocity component u increases with increase in parameter U. Figures **12-15** show effect of parameters m, K, U, D and E E on the velocity component v. These figures indicate that v increases with increase in these parameters in absolute value. These figures also indicate that increase in v is larger with increase in parameters K and m than

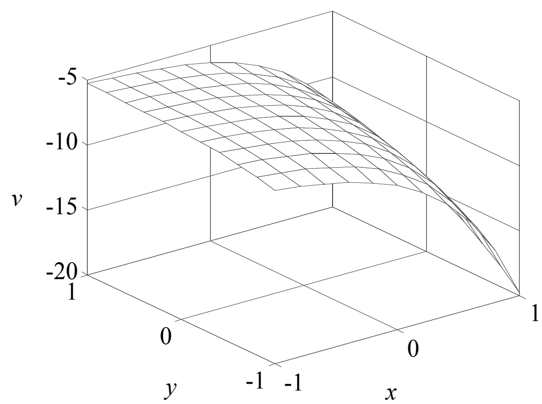

Figure 12: Variation in velocity component v in the flow field for U = 5, K = 3, m = 2, D = E = 1.

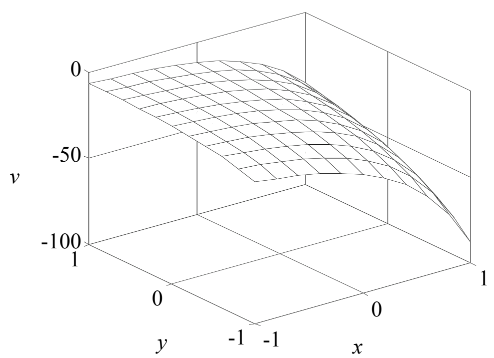

Figure 13: Variation in velocity component v in the flow field for U = 5, K = 3, m = 2, D = 5, E = 9.

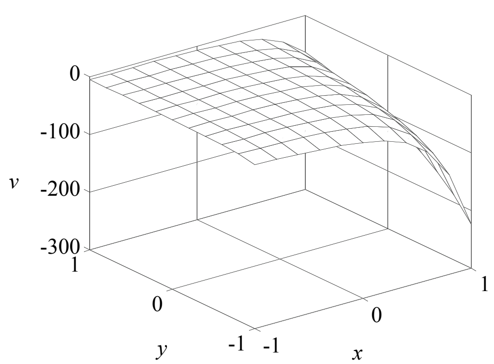

Figure 14: Variation in velocity component v in the flow field for U = 5, K = 12, m = 4, D = E = 1.

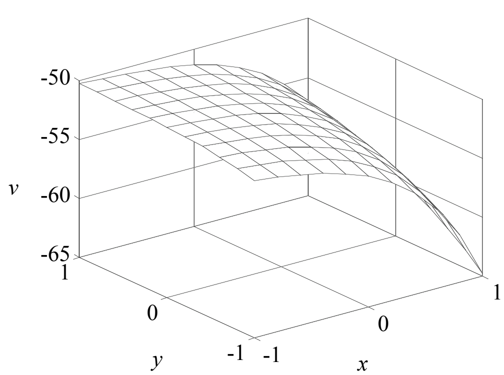

Figure 15: Variation in velocity component v in the flow field for U = 50, K = 3, m = 2, D = E = 1.

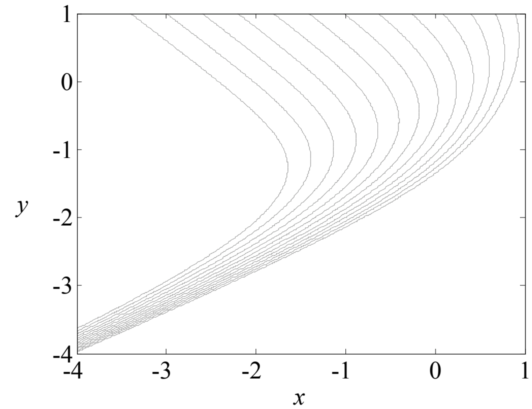

Figure 16: Streamlines for the streamfunction equation (44) for U = 5, m = 3, D = E = 1.

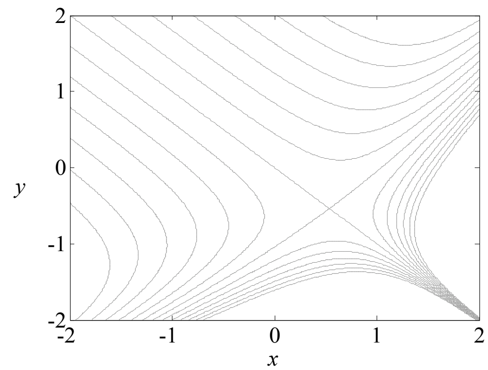

Figure 17: Streamlines for the streamfunction equation (44) for U = 5, K = 5, m = 3, D = –1, E = 1.

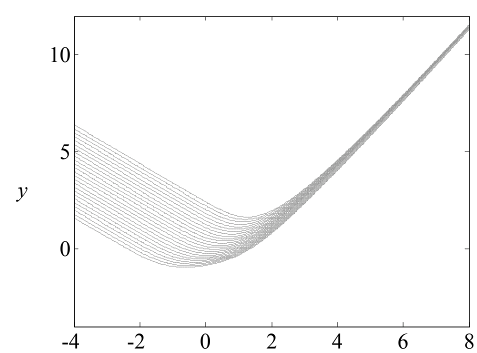

Figure 18: Streamlines for the streamfunction equation (44) for U = 5, K = 5, m = 3, D = -1, E = –1.

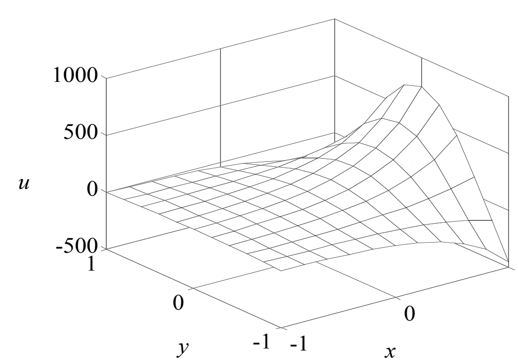

Figure 19: Variation in velocity component u in the flow field for U = 5, K = 3, m = 5, L = M = 1.

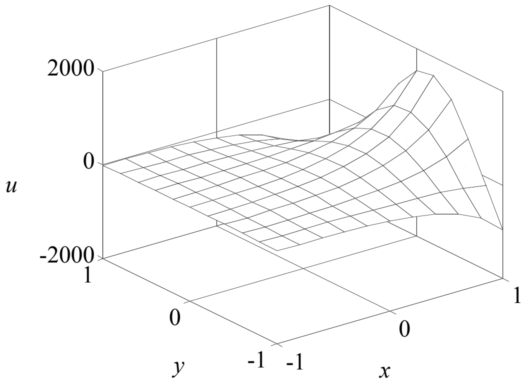

Figure 20: Variation in velocity component *u* in the flow field for *U* = 5, *K* = 5, *m* = 6, *L* = *M* = 1.

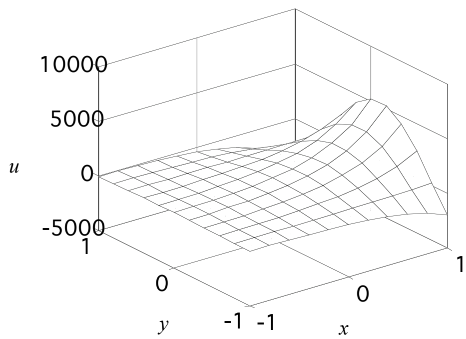

Figure 21: Variation in velocity component *u* in the flow field for *U* = 5, *K* = 53, *m* = 5, *L* = 5, *M* = 9.

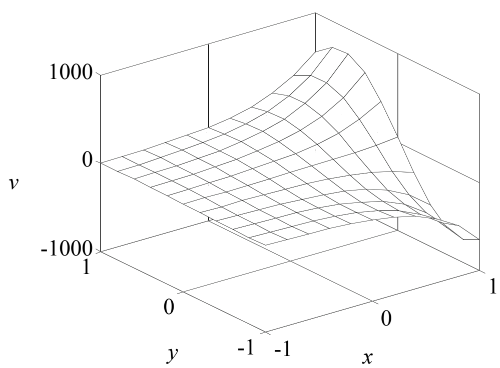

Figure 22: Variation in velocity component *v* in the flow field for *U* = 5, *K* = 3, *m* = 5, *L* = *M* = 1.

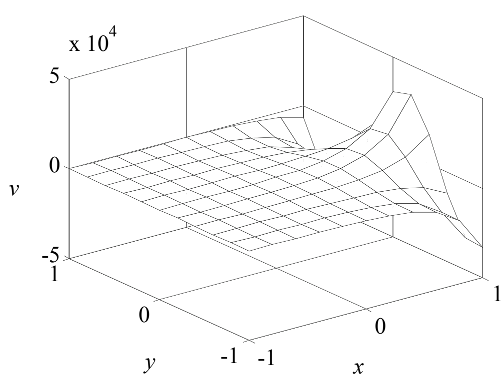

Figure 23: Variation in velocity component *v* in the flow field for *U* = 5, *K* = 8, *m* = 9, *L* = *M* = 1.

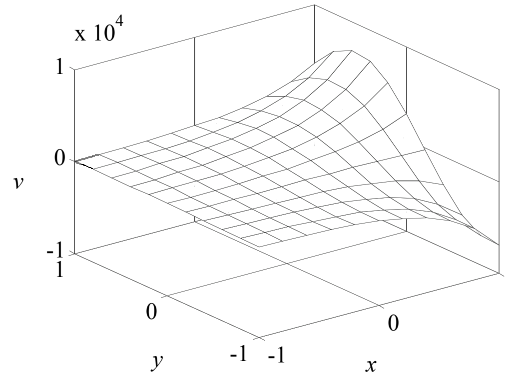

Figure 24: Variation in velocity component *v* in the flow field for *U* = 5, *K* = 3, *m* = 5, *L* = 6, *M* = 12.

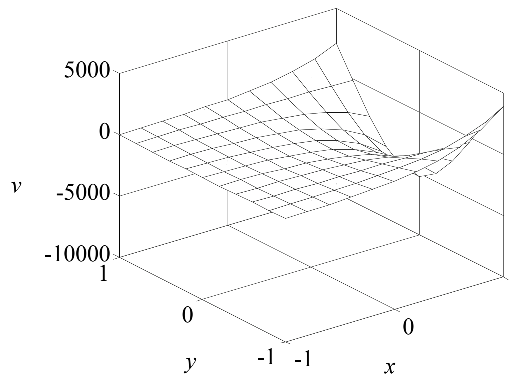

Figure 25: Variation in velocity component u in the flow field for *U* = 5, *K* = 3, *m* = 5, *L* = 5, *M* = -9.

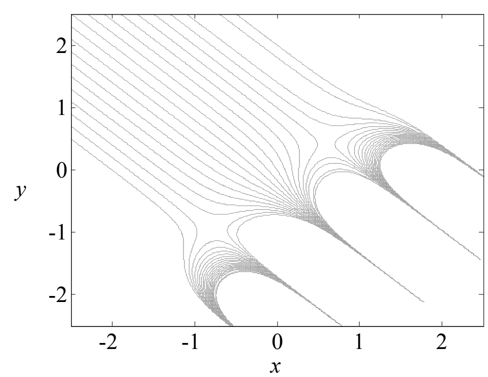

Figure 26: Streamlines for the streamfunction equation (48) for *U* = 5, *K* = 2, *m* = 5, *L* = 1, *M* = 1.

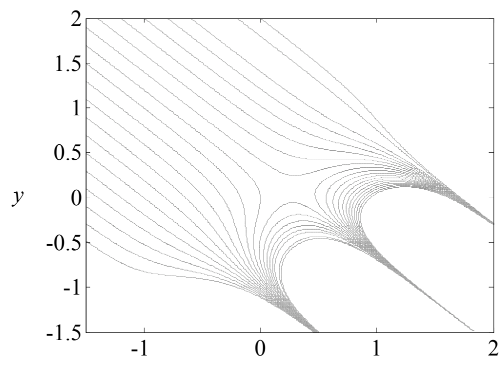

Figure 27: Streamlines for the streamfunction equation (48) for U = 5, K = 2, m = 5, L = 1, M = –1.

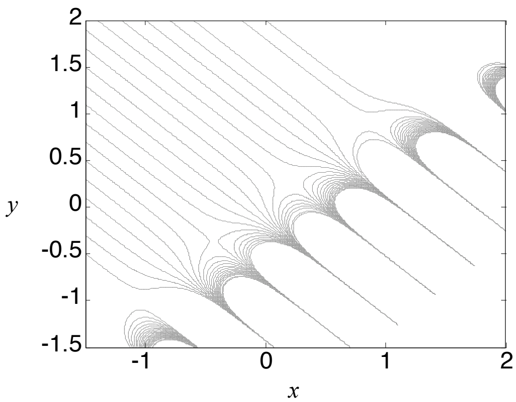

Figure 28: Streamlines for the streamfunction equation (48) for U = 5, K = 1, m = 10, L = 1, M = 1.

increase in other parameters. Figures **16-18** show the streamlines for case II, which illustrate the behaviour of the flows for the different values of the parameters *m, K, U, D* and *E*. Figures **19-25** show the effect of parameters *m, K, U, L* and *M* on the velocity components *u* and *v* for case III. These figures indicate that the values of the velocity components *u* and *v* variate in the flow region with change in parameters *m, K, U, L* and *M*. Figures **26-28** represent the streamlines for the streamfunction for case III for various values of the parameter *m, K, U, L* and *M*. Similarly we can discuss the effect of parameters *m, U, P* and *Q* on the velocity components *u* and *v* for case IV through their plots. We mention that influence of the parameters on velocity components *u* and *v* are of the same nature as in cases I to III.

6. CONCLUSION

A class of exact solutions to equations governing the steady motion of incompressible couple stress fluids are determined for which the vorticity distribution is given by the Eq. (12). The solutions are determined by introducing the function *H* and the canonical coordinates ξ, η defined by Eqs. (17) and (19). The effect of the pertinent parameters *m, K, U, A, B, D, E, L* and *M* on velocity components *u* and *v* are discussed and streamlines are presented for various values of parameters. It is found that velocity components *u*and *v* increase with increase in parameters *m, K, U, A, B, D, E, L* and *M*. The increase in velocity components *u* and *v* is much larger with increase in *K* and *m* than increase in other parameters.

REFERENCES

[1] Stokes K. Couple stresses in fluids. Physics Fluids. 1966; 9: 1709-15.
http://dx.doi.org/10.1063/1.1761925

[2] Mindlin RD, Tierstea HF. Effect of couple stresses in linear elasticity. Arch Rat Mech Anal. 1962; 11: 415-48.
http://dx.doi.org/10.1007/BF00253946

[3] Ramanaiah G. Squeeze films between finite plates lubricated by fluid with couple stresss. Wear. 1979; 54: 315-20.
http://dx.doi.org/10.1016/0043-1648(79)90123-6

[4] Sinha P, Singh C. Couple stress in the lubrication of rolling contact bearings considering cavitations. Wear. 1981; 67: 85-98.
http://dx.doi.org/10.1016/0043-1648(81)90077-6

[5] Gupta RS. Sharma LG. Analysis of couple stresses lubricant in hydrostatics thrust bearing. Wear. 1988; 125: 257-69.
http://dx.doi.org/10.1016/0043-1648(88)90117-2

[6] Wang L. Zhu Q, Zhu W. On the performance of dynamically loaded journal bearings with couple stress fluids. Tribiology Int. 2002: 35(3): 185-91.
http://dx.doi.org/10.1016/S0301-679X(01)00114-1

[7] Naduvinamant NB, Hiremath PS, Girubasavaraj G. Squeeze film lubrication of a short porous journal bearing with couple stress fluids. Tribiology. 2001; 34: 739-47.
http://dx.doi.org/10.1016/S0301-679X(01)00064-0

[8] Naduvinamant NB, Hiremath PS, Girubasavaraj G. Surface roughness effects in a short porous journal bearing with a couple stress fluid. Fluid Dynamics Reseach. 2002; 31: 333-54.
http://dx.doi.org/10.1016/S0169-5983(02)00137-5

[9] Naduvinamant NB, Hiremath PS, Girubasavaraj G. Effects of surface roughness on the couple stress squeeze film between a sphere and a flat plate. Tribiology Int. 2005; 38: 451-8.
http://dx.doi.org/10.1016/j.triboint.2004.09.001

[10] Islam S, Zhou CY. Exact solutions for two dimensional flows of couple stress fluids. ZAMP. 2007; 58: 1035-48.
http://dx.doi.org/10.1007/s00033-007-5075-5

[11] Devarkar M, Iyengar TKV. Stokes ' Problems for an incompressible couple stress fluid. Non linear Analysis: Modelling and Control. 2008; 1: 181-90.

[12] Devarkar M, Iyengar TKV. Run up flow of a Couple stress fluid between parallel plates. Nonlinear Analysis: Modelling and Control. 2010; 15: 29-37.

[13] Naeem RK. Some New exact solutions to unsteady flow equations of incompressible couple stress fluid. International Journal of Applied Mathematics and Mechanics. 2010; 6: 1-11.

Received on 01-10-2011 Accepted on 12-11-2011 Published on 24-01-2012

DOI: http://dx.doi.org/10.6000/1927-5129.2012.08.01.02