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Convergence Analysis for Linear Fredholm and Nonlinear Fredholm Hammerstein Integral Equations

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Abstract:

In this article, we consider the linear Fredholm integral equations and Fredholm-Hammerstein's integral equations. We propose the Legendre polynomial based degenerate kernel method to solve linear Fredholm and Fredholm-Hammerstein integral equations. We discuss the convergence and error analysis of the proposed method and also obtain the superconvergence results for iterated degenerate kernel method.

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1. INTRODUCTION

In this article, we consider the second kind Fredholm integral equation

$$x(t)-\int_{-1}^{1}k(t,s)x(s)ds=f(t),\ t\in[-1,1], \tag{1.1}$$

and the Hammerstein equation

$$x(t)-\int_{-1}^{1}k(t,s)\psi(s,x(s))ds=f(t),\ t\in[-1,1], \tag{1.2}$$

where $k(\cdot,\cdot)$ and $f(\cdot)$ are known functions in the integral equations (1.1) and (1.2) and $\psi(\cdot,x(\cdot))$ is known in the integral equation (1.2), and x is the unknown function in both the cases to be found in the Banach space $\mathbb{X}=C[-1,1]$. It is well known that there are numerous numerical techniques available to find the approximations to integral equation solutions, including the Galerkin, collocation, and Petrov Galerkin methods. [1] has found that the iterated version of Galerkin and collocation methods provide more accurate approximate solutions to the solution x than the Galerkin and collocation approximations. This iterated technique was also extended to Petrov Galerkin methods, discrete Petrov Galerkin methods, degenerate kernel methods and new projection methods (see [2-8]). Lardy in [9] presented an alternative to Nyström's method. Kumar and Sloan [6] proposed a new type collocation methods and its superconvergence properties were studied by Kumar [10]. Han [11] also offered an extrapolation of a discrete form of a collocation-type method. Brunner [12] talked about the relationship between Kumar and Sloan's method and the iterated spline collocation method for Hammerstein equations. Kaneko and Xu [13] devised a degenerated kernel approach for Hammerstein equations. Kaneko and Xu [14] established the superconvergence of the iterated Galerkin solutions for the Hammerstein equations with smooth and weakly singular kernels. Kaneko and Xu [2] established the superconvergence of the iterated Galerkin solutions for the Hammerstein equations with smooth and weakly singular kernels.

In this article, we consider the degenerate kernel method, which plays an important role in the study of the second kind Fredholm integral equations. Let K and $K\Psi$ be the compact linear and non-linear integral operators. We assume that 1 is not an eigenvalue of K and $K\Psi$. The degenerate kernel method for approximating the solutions of the equations (1.1) and (1.2) consist of replacing the kernel by the finite rank approximation. In particular

$$k_n(t,s)=\sum_{i=1}^{n}A_i(t)B_i(s), \tag{1.3}$$

where A and B are in $\mathbb{X}$. The approximate solution of the integral equation (1.1) and (1.5) are given by

$$x_n(t)-\int_{-1}^{1}k_n(t,s)x(s)ds=f(t), \tag{1.4}$$

and

$$x_n(t)-\int_{-1}^{1}k_n(t,s)\psi(s,x(s))ds=f(t). \tag{1.5}$$

2. LEGENDRE DEGENERATE KERNEL METHOD FOR LINEAR FREDHOLM INTEGRAL EQUATION

In this section, we discuss the degenerate kernel method and iterated degenerate kernel method for linear Fredholm integral equation based on Legendre polynomial basis functions and obtain the convergence results. Let $C[-1,1]$ be the Banach space of all continuous functions defined on $[-1,1]$ with the uniform norm. Consider the following linear integral equation:

$$x(t)-\int_{-1}^{1}k(t,s)x(s)ds=f(t),\ t\in[-1,1]. \tag{2.1}$$

Let

$$Kx(t)=\int_{-1}^{1}k(t,s)x(s)ds,\ t\in[-1,1]. \tag{2.2}$$

Then the integral equation (2.1) can be written in operator form

$$x(t)-Kx(t)=f(t),\ t\in[-1,1]. \tag{2.3}$$

Let $\mathbb{X}_n=Span\{\varphi_0,\varphi_1,\ldots,\varphi_n\}$ be the sequence of Legendre polynomials subspace of $\mathbb{X}$ of degree r and $\mathbb{Y}_n=Span\{\phi_0,\phi_1,\ldots,\phi_n\}$ be the sequence of Legendre polynomials subspace of $\mathbb{X}$ of degree r. Where $\{\varphi_0,\varphi_1,\ldots,\varphi_n\}$ and $\{\phi_0,\phi_1,\ldots,\phi_n\}$ form orthogonal basis for $\mathbb{X}_n$ and $\mathbb{Y}_n$, respectively. Here $\{\varphi_i\}$ and $\{\phi_j\}$ are given by

$$\varphi_i(s)=\sqrt{\frac{2i+1}{2}}L_i(s),\phi_j(s)=\sqrt{\frac{2j+1}{2}}L_j(s),\ i,j=0,1,\ldots,n,$$

here L_i's are the Legendre polynomial of degree $\leq i$. These Legendre polynomials can be generated by the following three-term recurrence relation

$$L_0(s)=1,\ L_1(s)=s,\ \ s\in[-1,1],$$

and

$$(i+1)L_{i+1}(s)=(2i+1)sL_i(s)-iL_{i-1}(s), i=1,2,\ldots,n-1.$$

Let L_x be a projection of $C([-1,1]\times[-1,1])$ onto $\mathbb{X}_n\times C([-1,1])$, where $\mathbb{X}_n$ is an n-dimensional subspace of $C[-1,1]$. Similarly, let M_y be a projection of $C([-1,1]\times[-1,1])$ onto $C([-1,1])\times\mathbb{Y}_n$, where $\mathbb{Y}_n$ is an n-dimensional subspace of $C[-1,1]$.

Define

$$R_x k(x,y) = k(x,y) - L_x k(x,y), \tag{2.4}$$

and

$$R_y k(x,y) = k(x,y) - M_y k(x,y). \tag{2.5}$$

Then $L_x k$ and $M_y k$ provide two degenerate kernels that approximate partially $k(x,y)$ with respect to x and y, with respective errors $R_x k$ and $R_y k$. Then $L_x M_y k$ gives a complete approximation of $k(x,y)$ in both x and y, with error term

$$R(L_x M_y)k = R_x k + R_y k - R_x R_y k. \tag{2.6}$$

Thus, the rate of convergence depends on the approximation powers of L_x and M_y.

In order to enhance the speed of convergence, we define the Boolean sum of L_x and M_y by

$$L_x \otimes M_y = L_x + M_y - L_x M_y. \tag{2.7}$$

Then $(L_x \otimes M_y)k$ approximates k with the error

$$R(L_x \otimes M_y)k = R_x R_y k. \tag{2.8}$$

Suppose that for each $n \geq 1$, $k_n(t,s)$ is an approximation of the kernel $k(t,s)$, and it's degenerate kernel form

$$k_n(t,s) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varphi_i(t)\varphi_j(s), \tag{2.9}$$

where $\{\varphi_i\}_{i=1}^{n}$ is a set of linearly independent functions in $C[-1,1]$.

Let

$$K_n x(t) = \int_{-1}^{1} k_n(t,s)x(s)ds. \tag{2.10}$$

The approximate solution x_n of the integral equation (2.1) can be found by solving the following approximation equation

$$x_n - K_n x_n = f. \tag{2.11}$$

Substituting k_n from the equation (3.4) in the equation (2.11), we obtain

$$x_n(t) - \int_{-1}^{1}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varphi_i(t)\varphi_j(s)x_n(s)ds = f(t)$$

$$x_n(t) = \sum_{i=1}^{n}\varphi_i(t)\{\sum_{j=1}^{n}\int_{-1}^{1} a_{ij}\varphi_j(s)x_n(s)ds\} + f(t). \tag{2.12}$$

Suppose

$$c_i = \sum_{j=1}^{n}\int_{-1}^{1} a_{ij}\varphi_j(s)x_n(s)ds. \tag{2.13}$$

Then x_n can be written as

$$x_n(t) = f(t) + \sum_{i=1}^{n} c_i\varphi_i(t). \tag{2.14}$$

Substituting equation (2.14) in the equation (2.13), we obtain

$$c_i = \sum_{j=1}^{n}\int_{-1}^{1} a_{ij}\varphi_j(s)\{f(s) + \sum_{l=1}^{n} c_l\varphi_l(s)\}ds$$

$$c_i - \sum_{l=1}^{n} c_l \sum_{j=1}^{n}\int_{-1}^{1} a_{ij}\varphi_j(s)\varphi_l(s)ds = \sum_{j=1}^{n}\int_{-1}^{1} a_{ij}\varphi_j(s)f(s)ds, 1 \leq i \leq n. \tag{2.15}$$

Once we get the $c_i's$ from (2.15), we obtain the desired approximate solution x_n from (2.14).

We define the iterated approximation corresponding to the equation (2.1) by

$$\tilde{x}_n = f + Kx_n. \tag{2.16}$$

lemma 1 Let $k(\cdot,\cdot) \in C^{(r,r)}(0,1)$ be the kernel of the integral equation (2.1) and the integral equation K be defined by (2.2). Let the operator K_n be defined by (2.10). Then there holds

$$\|(K - K_n)\|_\infty = \begin{cases} O(n^{-r}), \ if \ k_n = L_x M_y k, \\ O(n^{-2r}), if \ k_n = (L_x \otimes M_y)k. \end{cases} \tag{2.17}$$

Proof. Consider

$$\|(K - K_n)x\|_\infty = \sup_{s\in[-1,1]} |(K - K_n)x(s)|. \tag{2.18}$$

Let

$$|(K - K_n)x(s)| = |\int_{-1}^{1}[k_n(t,s) - k(t,s)]x(s)ds| \tag{2.19}$$

Case-I:- Let $k_n(t,s) = L_t M_s k(t,s)$, we obtain

$$|(K-K_n)x(s)|=|\int_{-1}^{1}[L_tM_s-I]k(t,s)x(s)ds|$$

$$\leq \|[L_tM_s-I]k(\cdot,\cdot)\|_{L^2}\|x\|_{L^2} \leq Cn^{-r}\|x\|_{L^2} \quad (2.20)$$

This implies

$$\|(K-K_n)x\|_\infty \leq Cn^{-r}\|x\|_{L^2}. \quad (2.21)$$

Case-II:- Let $k_n(t,s)=(L_t\otimes M_s)k(t,s)$, we obtain

$$|(K-K_n)x(s)|=|\int_{-1}^{1}[L_t\otimes M_s-I]k(t,s)x(s)ds|$$

$$=|\int_{-1}^{1}[(L_t+M_s-L_tM_s)-I]k(t,s)x(s)ds|$$

$$\leq \|(L_t+M_s-L_tM_s-I)k(\cdot,\cdot)\|_{L^2}\|x\|_{L^2}$$

$$\leq \|(I-L_t)(I-M_s)k(\cdot,\cdot)\|_{L^2}\|x\|_{L^2}$$

$$\leq Cn^{-2r}\|k^{(r,r)}(\cdot,\cdot)\|_{L^2}\|x\|_{L^2}. \quad (2.22)$$

This implies

$$\|(K-K_n)x\|_\infty \leq Cn^{-2r}\|x\|_{L^2}. \quad (2.23)$$

This completes the proof.

Theorem 1 Let the integral operator K be defined by (2.2) and the integral operator K_n be defined by (2.10). Suppose that 1 is not an eigenvalue of K, then ∃ a finite constant $M>0$ such that $\|(I-K_n)^{-1}\|_\infty < M < \infty$.

Proof. We need to show that K_n is norm convergent to K in the uniform norm.

From Lemma (1), we have

$$\|(K-K_n)\|_\infty = \begin{cases} O(n^{-r}), & if\ k_n = L_xM_yk, \\ O(n^{-2r}), & if\ k_n=(L_x\otimes M_y)k. \end{cases} \quad (2.24)$$

This implies $\|K-K_n\|_\infty \to 0$ as $n\to\infty$. Since 1 is not an eigenvalue of K. Hence from the analysis of [15], $\exists\, M>0$ such that $\|(I-K_n)^{-1}\|_\infty < M$.

This completes the proof.

In the following theorem, we discuss the convergence analysis of degenerate kernel method.

Theorem 2 Let x be the exact solution of the integral equation (2.1) and the approximate solution x_n be defined by (2.11), then we find the following error bounds in the degenerate kernel method

$$\|x-x_n\|_\infty = \begin{cases} O(n^{-r}), & if\ k_n = L_xM_yk, \\ O(n^{-2r}), & if\ k_n=(L_x\otimes M_y)k. \end{cases} \quad (2.25)$$

Proof. From equations (2.3) and (2.11), we have

$$x-x_n=(I-K)^{-1}f-(I-K_n)^{-1}f$$

$$=(I-K_n)^{-1}[I-K_n-I-K](I-K)^{-1}f$$

$$=(I-K_n)^{-1}(K-K_n)x. \quad (2.26)$$

Hence

$$\|x-x_n\|_\infty \leq \|(I-K_n)^{-1}\|_\infty \|(K-K_n)x\|_\infty. \quad (2.27)$$

Now using the Lemma 1 and Theorem 1 in the estimate (2.27), we obtain

$$\|x-x_n\|_\infty \leq M\|(K-K_n)x\|_\infty = \begin{cases} O(n^{-r}), & if\ k_n = L_xM_yk, \\ O(n^{-2r}), & if\ k_n=(L_x\otimes M_y)k. \end{cases} \quad (2.28)$$

This completes the proof.

In the next theorem, we discuss the order of convergence in iterated degenerate kernel method.

Theorem 3 Let x be the exact solution of the integral equation (2.1) and the approximate solution $\tilde{x}_n$ be defined by (2.16), then we find the following error bounds in the iterated degenerate kernel method

$$\|x-\tilde{x}_n\|_\infty = \begin{cases} O(n^{-2r}), & if\ k_n = L_xM_yk, \\ O(n^{-4r}), & if\ k_n=(L_x\otimes M_y)k. \end{cases} \quad (2.29)$$

Proof. From equations (2.3) and (2.16), we have

$$x-\tilde{x}_n=K(x-x_n)$$

$$=K(I-K)^{-1}(K-K_n)x_n$$

$$\|x-\tilde{x}_n\|_\infty \leq \|(I-K)^{-1}\|_\infty \|K(K-K_n)x_n\|_\infty \quad (2.30)$$

Now we evaluate the error bounds for $K(K-K_n)x_n$ in the uniform norm.

Consider

$$|K(K-K_n)x_n(s)|=|\int_{-1}^{1}k(t,s)(K-K_n)x_n(s)ds|$$

$$= \left| \int_{-1}^{1} k(t,s) \int_{-1}^{1} [k(s,\xi) - k_n(s,\xi)] x_n(s) d\xi ds \right|$$

$$= \left| \int_{-1}^{1} \int_{-1}^{1} k(t,s) [k(s,\xi) - k_n(s,\xi)] x_n(s) d\xi ds \right|$$

Let $\varphi_t(t,s) = k(t,s)x_n(s)$ and let $\psi_n(s,\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \varphi_i(s) \phi_j(\xi)$ be any element in $X_n \times Y_n$. Then since k_n is the least-squares approximation of k,

$$\int_{-1}^{1} \int_{-1}^{1} \psi_n(s,\xi)[k(t,s) - k_n(t,s)] ds d\xi = 0, \tag{2.31}$$

therefore

$$| K(K - K_n)x_n(s) | = \left| \int_{-1}^{1} \int_{-1}^{1} [\varphi_t(s,\xi) - \psi_n(s,\xi)] [k(s,\xi) - k_n(s,\xi)] d\xi ds \right|$$

$$\leq \left\| \varphi_t(s,\xi) - \psi_n(s,\xi) \right\|_{L^2} \left\| k(s,\xi) - k_n(s,\xi) \right\|_{L^2}$$

$$= \begin{cases} O(n^{-2r}), \ if \ k_n = L_x M_y k, \\ O(n^{-4r}), \ if \ k_n = (L_x \otimes M_y)k. \end{cases} \tag{2.32}$$

Using the estimate (2.31) in the estimate (2.30), we have

$$\left\| x - \tilde{x}_n \right\|_\infty = \begin{cases} O(n^{-2r}), \ if \ k_n = L_x M_y k, \\ O(n^{-4r}), \ if \ k_n = (L_x \otimes M_y)k. \end{cases} \tag{2.33}$$

This completes the proof.

Remark 1 In the above section, we have discussed the degenerate kernel method and iterated degenerate kernel method in the two cases when $k_n = L_x M_y k$ and $k_n = (L_x \otimes M_y)k$. From Theorems 2 and 3, we have seen that the order of convergence in iterated degenerate kernel method improves over degenerate kernel method.

3. LEGENDRE DEGENERATE KERNEL METHOD FOR FREDHOLM-HAMMERSTEIN INTEGRAL EQUATION

In this section, we discuss the degenerate kernel method and it's iterated version to obtain the error analysis for Fredholm-Hammerstein integral equations. Consider the following Fredholm-Hammerstein integral equation:

$$x(t) - \int_{-1}^{1} k(t,s) \psi(s, x(s)) ds = g(t), \ t \in [-1,1]. \tag{3.1}$$

Let

$$K\Psi(x)(t) = \int_{-1}^{1} k(t,s) \psi(s, x(s)) ds, \ t \in [-1,1]. \tag{3.2}$$

Then the integral equation (3.1) can be written in operator form

$$x(t) - K\Psi(x)(t) = g(t). \tag{3.3}$$

Suppose that for each $n \geq 1$, $k_n(t,s)$ is an approximation of the kernel $k(t,s)$, and it's degenerate kernel form

$$k_n(t,s) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \varphi_i(t) \varphi_j(s), \tag{3.4}$$

where $\{\varphi_i\}_{i=1}^{n}$ is a set of linearly independent functions in $C[-1,1]$.

Let

$$K_n \Psi(x)(t) = \int_{-1}^{1} k_n(t,s) \psi(s, x(s)) ds. \tag{3.5}$$

Let x_n be the approximating solution of the integral equation (3.1). We denote the approximating equation by

$$x_n - K_n \Psi(x_n) = g, \tag{3.6}$$

The equation that one must solve is the following

$$x_n(t) - \int_{-1}^{1} k_n(t,s) \psi(s, x_n(s)) ds = g(t), \ -1 \leq t \leq 1. \tag{3.7}$$

Following analogously the development made in (3.4) and (2.12) with

$$c_i = \sum_{j=1}^{n} \int_{-1}^{1} a_{ij} \varphi_j(s) \psi(s, x_n(s)) ds. \tag{3.8}$$

x_n can be written as

$$x_n(t) = g(t) + \sum_{i=1}^{n} c_i \varphi_i(t). \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain the following n nonlinear equations in n unknowns $c_1, c_2, \ldots, c_n$.

$$c_i = \sum_{j=1}^{n} \int_{-1}^{1} a_{ij} \varphi_j(s) \psi(s, g(s) + \sum_{l=1}^{n} c_l \varphi_l(s)) ds, 1 \leq i \leq n. \tag{3.10}$$

We define the iterated approximation corresponding to the equation (3.1) by

$$\tilde{x}_n = g + K\Psi(x_n). \quad (3.11)$$

In the following theorem, we discuss the error analysis in degenerate kernel method.

Theorem 4 Let x be the exact solution of the integral equation (3.1) and x_n be the approximate solution of the equation (3.6). Then the following holds

$$\left\|x - x_n\right\|_\infty = \begin{cases} O(n^{-r}), \text{ if } k_n = L_x M_y k, \\ O(n^{-2r}), \text{ if } k_n = (L_x \otimes M_y)k. \end{cases} \quad (3.12)$$

Proof. First we need to show that $\left\|(I-(K_n\Psi)'(x))^{-1}\right\|_\infty$ exists and bounded.

For this, we need to show that $\left\|(K_n\Psi)'(x)) - (K\Psi)'(x))\right\|_\infty \to 0$ as $n \to \infty$.

Consider

$$(K_n\Psi)'(x)) - (K\Psi)'(x)) = \int_{-1}^{1}[k_n(t,s)\psi^{(0,1)}(s,x(s)) - k(t,s)\psi^{(0,1)}(s,x(s))]ds$$

$$= \int_{-1}^{1}[k_n(t,s) - k(t,s)]\psi^{(0,1)}(s,x(s))ds$$

$$\leq \left\|k_n(t,\cdot) - k(t,\cdot)\right\|_{L^2}\left\|\psi^{(0,1)}(\cdot,x(\cdot))\right\|_{L^2}. \quad (3.13)$$

Since k_n is the least square approximation of k. Thus we have

$$\left\|(K_n\Psi)'(x)) - (K\Psi)'(x))\right\|_\infty \to 0 \text{ as } n \to \infty. \quad (3.14)$$

Since 1 is not eigenvalue of $(K\Psi)'(x)$ and from [15], there exists $\mathcal{L}_1 > 0$ such that $\left\|(I-(K_n\Psi)'(x))^{-1}\right\|_\infty < \mathcal{L}_1$.

From equations (3.3) and (3.6), we have

$$x_n - x = (K_n\Psi)(x_n) - (K\Psi)(x)$$

$$= (K_n\Psi)(x_n) - (K_n\Psi)(x) + (K_n\Psi)(x) - (K\Psi)(x)$$

$$= (K_n\Psi)(x_n) - (K_n\Psi)(x) - (K_n\Psi)'(x)(x_n - x) + (K_n\Psi)'(x)(x_n - x) + (K_n\Psi)(x) - (K\Psi)(x).$$

This implies

$$(I-(K_n\Psi)'(x))(x_n - x) = (K_n\Psi)(x_n) - (K_n\Psi)(x) - (K_n\Psi)'(x)(x_n - x) + (K_n\Psi)(x) - (K\Psi)(x).$$

Hence

$$x_n = x + (I-(K_n\Psi)'(x))^{-1}[(K_n\Psi)(x_n) - (K_n\Psi)(x) - (K_n\Psi)'(x)(x_n - x) + (K_n\Psi)(x) - (K\Psi)(x)]$$

$$= A_n(x_n),$$

where

$$A_n(v) = x + (I-(K_n\Psi)'(x))^{-1}[(K_n\Psi)(v) - (K_n\Psi)(x) - (K_n\Psi)'(x)(v - x) + (K_n\Psi)(x) - (K\Psi)(x)]. \quad (3.15)$$

From above, it can be seen that $x_n = (K_n\Psi)(x_n)$ iff $A_n(x_n) = x_n$. Next we show that $A_n : B(x,\delta) \to B(x,\delta)$, for some $\delta > 0$ is a contraction mapping.

Now to prove $A_n(B(x,\delta)) \subseteq B(x,\delta)$. For this, $\forall v \in B(x,\delta)$ and from equation (3.15), we obtain

$$A_n(v) - x = (I-(K_n\Psi)'(x))^{-1}[(K_n\Psi)(v) - (K_n\Psi)(x) - (K_n\Psi)'(x)(v - x) + (K_n\Psi)(x) - (K\Psi)(x)]$$

$$= (I-(K_n\Psi)'(x))^{-1}[\{(K_n\Psi)'(v + \theta_1(v - x)) - (K_n\Psi)'(x)\}(v - x) + (K_n\Psi)(x) - (K\Psi)(x)]. \quad (3.16)$$

Since $y_1 = v + \theta_1(v - x) \in B(x,\delta), 0 < \theta_1 < 1$ and using the estimate (3.16) and Mean value theorem, we obtain

$$\| A_n(v) - x \|_\infty$$

$$\leq \| (I-(K_n\Psi)'(x))^{-1}[\{(K_n\Psi)'(y_1) - (K_n\Psi)'(x)\}(v - x)] \|_\infty + \| (I-(K_n\Psi)'(x))^{-1}[(K_n\Psi)(x) - (K\Psi)(x)] \|_\infty$$

$$\leq \mathcal{L}C_2 \| y_1 - x \|_\infty \| v - x \|_\infty + \mathcal{L}n^{-2r}$$

$$\leq \mathcal{L}C_2 \| v - x \|_\infty^2 + \mathcal{L}n^{-2r}. \quad (3.17)$$

Since $n^{-2r} \to 0$ as $n \to \infty$, we select n sufficient large so that $n^{-2r} < \delta^2$ and δ small enough such that $(\mathcal{L}(C_2 + 1)\delta) < 1$, then from equation (3.17), we obtain

$$\| A_n(v) - x \|_\infty \leq (\mathcal{L}C_2\delta + \mathcal{L}\delta)\delta < \delta. \quad (3.18)$$

It follows $A_n(B(x,\delta)) \subseteq B(x,\delta)$.

Next we show that A_n is a contraction mapping. Let for any $\eta_1, \eta_2 \in B(x,\delta)$ and using the estimate (3.16), we obtain

$$\| A_n(\eta_1) - A_n(\eta_2) \|_\infty$$

$$= \| (I-(K_n\Psi)'(x))^{-1}[(K_n\Psi)(\eta_1) - (K_n\Psi)'(x)(\eta_1 - x) - (K_n\Psi)(\eta_2) + (K_n\Psi)'(x)(\eta_2 - x)] \|_\infty$$

$$\leq \mathcal{L}\,\|(K_n\Psi)(\eta_1)-(K_n\Psi)(\eta_2)-(K_n\Psi)'(x)(\eta_1-\eta_2)\|_\infty$$

$$=\mathcal{L}\,\|(K_n\Psi)'(\eta_1+\theta_1(\eta_1-\eta_2))(\eta_1-\eta_2)-(K_n\Psi)'(x)(\eta_1-\eta_2)\|_\infty$$

$$\leq \mathcal{L}\,\|(K_n\Psi)'(\eta_1+\theta_1(\eta_1-\eta_2))-(K_n\Psi)'(x)\|_\infty\|\eta_1-\eta_2\|_\infty$$

$$\leq \mathcal{L}C_2\delta\|\eta_1-\eta_2\|_\infty .$$

We select δ very small such that $\mathcal{L}C_2\delta<1$, it follows

$$\|A_n(\eta_1)-A_n(\eta_2)\|_\infty \leq \|\eta_1-\eta_2\|_\infty . \tag{3.19}$$

It shows that $A_n : B(x,\delta)\to B(x,\delta)$ is a contraction mapping. Hence from Banach contraction principal, A_n has an isolated solution x_n in $B(x,\delta)$.

From equations (3.3) and (3.6), we obtain

$$x(t)-x_n(t)=K\Psi(x)(t)-K_n\Psi(x_n)(t)$$

$$=\int_{-1}^{1}[k(t,s)\psi(s,x(s))-k_n(t,s)\psi(s,x_n(s))]ds$$

$$\begin{aligned}&=\int_{-1}^{1}[k(t,s)-k_n(t,s)]\psi(s,x(s))ds+\\&\int_{-1}^{1}[\psi(s,x(s))-\psi(s,x_n(s))]k_n(t,s)ds\end{aligned} \tag{3.20}$$

This implies

$$\|x-x_n\|_\infty \leq \|k-k_n\|_{L^2}\|\psi\|_{L^2}+\sqrt{2}\|x-x_n\|_\infty\|k_n\|_{L^2}$$

$$(1-\sqrt{2}B)\|x-x_n\|_\infty \leq \|k-k_n\|_{L^2}\|\psi\|_{L^2} \tag{3.21}$$

Hence using Theorem 1, we obtain

$$\|x-x_n\|_\infty \leq \begin{cases} Cn^{-r}, & \text{if } k_n=L_xM_yk,\\ Cn^{-2r}, & \text{if } k_n=(L_x\otimes M_y)k\end{cases}$$

$$=\begin{cases} O(n^{-r}), & \text{if } k_n=L_xM_yk,\\ O(n^{-2r}), & \text{if } k_n=(L_x\otimes M_y)k.\end{cases} \tag{3.22}$$

This completes the proof.

In the following theorem, we discuss the error bounds in iterated degenerate kernel method.

Theorem 5 Let x be the exact solution of the integral equation (3.1) and $\tilde{x}_n$ denote the iterated approximate solution of the equation (3.11) in iterated degenerate kernel method. Then the following holds

$$\|x-\tilde{x}_n\|_\infty = \begin{cases} O(n^{-2r}), & \text{if } k_n=L_xM_yk,\\ O(n^{-4r}), & \text{if } k_n=(L_x\otimes M_y)k.\end{cases} \tag{3.23}$$

Proof. From equations (3.3) and (3.11) and using Mean value theorem, we obtain

$$x-\tilde{x}_n=K\Psi(x)-K\Psi(x_n)$$

$$=(K\Psi)'(x+\theta(x_n-x))(x_n-x)$$

$$=[(K\Psi)'(x+\theta(x_n-x))-(K\Psi)'(x)+(K\Psi)'(x)](x_n-x), \tag{3.24}$$

where $0<\theta<1$.

Hence using the assumption , we obtain

$$\begin{aligned}\|x-\tilde{x}_n\|_\infty &\leq \|[(K\Psi)'(x+\theta(x_n-x))-(K\Psi)'(x)](K\Psi)'(x)(x_n-x)\|_\infty\\&+\|(K\Psi)'(x)(x_n-x)\|_\infty\end{aligned}$$

$$\leq \|\theta(x_n-x)\|_\infty\|x_n-x\|_\infty+\|(K\Psi)'(x)(x_n-x)\|_\infty$$

$$\leq \|x_n-x\|_\infty^2+\|(K\Psi)'(x)(x_n-x)\|_\infty . \tag{3.25}$$

Again from equations (3.3) and (3.6), we obtain

$$x_n-x=K_n\Psi(x_n)-K\Psi(x)$$

$$=K_n\Psi(x_n)-K_n\Psi(x)-(K_n\Psi)'(x)(x_n-x)+(K_n\Psi)'(x)(x_n-x)$$

$$+K_n\Psi(x)-K\Psi(x)$$

$$(x_n-x)=K_n\Psi(x_n)-K_n\Psi(x)-(K_n\Psi)'(x)(x_n-x)+K_n\Psi(x)-K\Psi(x)$$

$$=[(K_n\Psi)'(x+\theta(x_n-x))-(K_n\Psi)'(x)](x_n-x)+[K_n\Psi(x)-K\Psi(x)]$$

$$x_n-x=[I-(K_n\Psi)'(x)]^{-1}[(K_n\Psi)'(x+\theta(x_n-x))-(K_n\Psi)'(x)](x_n-x)$$

$$+[I-(K_n\Psi)'(x)]^{-1}[(K_n\Psi)(x)-(K\Psi)(x)] \tag{3.26}$$

From estimate (3.24), we obtain

$$\|(K\Psi)'(x)(x_n-x)\|_\infty \leq c_2\|(K\Psi)'(x)[I-(K_n\Psi)'(x)]^{-1}\|_\infty\|x-x_n\|_\infty^2$$

$$+\|(K\Psi)'(x)[I-(K_n\Psi)'(x)]^{-1}[K_n\Psi(x)-K\Psi(x)]\|_\infty$$

$$\leq M\|x-x_n\|_\infty^2+\left\|\begin{matrix}(K\Psi)'(x)\{I+[I-(K_n\Psi)'(x)]^{-1}\\(K_n\Psi)'(x)\}[K_n\Psi(x)-K\Psi(x)]\end{matrix}\right\|_\infty$$

$$\leq M\|x-x_n\|_\infty^2+\|(K\Psi)'(x)[K_n-K]\Psi(x)\|_\infty$$

$$+\|(K\Psi)'(x)[I-(K_n\Psi)'(x)]^{-1}(K_n\Psi)'(x)[(K_n\Psi)(x)-(K\Psi)(x)]\|_\infty . \tag{3.27}$$

Consider

$$|(K_n\Psi)'(x)[(K_n\Psi)(x)-(K\Psi)(x)|=|$$
$$\int_{-1}^{1}k(t,s)\psi^{(0,1)}(s,x(s))(K_n-K)\Psi(x)(s)ds|$$

$$=|\int_{-1}^{1}k(t,s)\psi^{(0,1)}(s,x(s))\int_{-1}^{1}[k_n(s,\xi)-k(s,\xi)]\psi(\xi,x(\xi))d\xi ds|$$

$$=|\int_{-1}^{1}\int_{-1}^{1}k(t,s)\psi^{(0,1)}(s,x(s))[k_n(s,\xi)-k(s,\xi)]\psi(\xi,x(\xi))d\xi ds|$$

$$=|\int_{-1}^{1}\int_{-1}^{1}[\psi_t(s,\xi)-\varphi_n(s,\xi)][k_n(s,\xi)-k(s,\xi)]\psi(\xi,x(\xi))d\xi ds|$$

$$\leq \left\|\psi_t(s,\xi)-\varphi_n(s,\xi)\right\|_{L^2}\left\|k_n(s,\xi)-k(s,\xi)\right\|_{L^2}$$

$$=\begin{cases} O(n^{-2r}), \ if \ k_n = L_x M_y k, \\ O(n^{-4r}), if \ k_n = (L_x \otimes M_y)k. \end{cases} \tag{3.28}$$

Hence combining the estimates (3.26)-(3.28) with the estimate (3.25), we obatin

$$\left\|x-\tilde{x}_n\right\|_\infty = \begin{cases} O(n^{-2r}), \ if \ k_n = L_x M_y k, \\ O(n^{-4r}), if \ k_n = (L_x \otimes M_y)k. \end{cases} \tag{3.29}$$

This completes the proof.

Remark 2 In the above section, we have discussed the degenerate kernel method and iterated degenerate kernel method for Fredholm-Hammerstein integral equations and obtained the error analysis. From Theorems 4 and 5, we have seen that iterated degenerate kernel method improves over degenerate kernel method.

4. CONCLUSION

In this paper, we have discussed the degenerate kernel method and iterated degenerate kernel method for linear Fredholm integral equations and Fredholm-Hammerstein integral equations and obtained the error analysis. We have seen that iterated degenerate kernel method improves over degenerate kernel method in both the types of integral equations.

CONTRIBUTION OF THE AUTHORS

1st author- The author carried out the convergence analysis and proposed method to solve the problem.

2nd author- The author also contributes in convergence analysis.

3rd author- The author has proposed the problem and provide the technical details.

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