# A Validation of the Real Zeros of the Riemann Zeta Function via the Continuation Formula of the Zeta Function

O.O.A. Enoch[1] and F.J. Adeyeye[2]

[1]*Department of Mathematical Sciences, Ekiti State University, Ado-Ekiti, Nigeria*

[2]*Department of Math/Computer, College of Sciences, Federal University of Petroleum Resources, Effurun, Delta State*

**Abstract:** In this paper, the analytic continuation formula of the Riemann zeta function is presented as a function of $t^{2n}$, thus validating Riemann's claim that $\varepsilon(t)$ allows itself to be developed in the power of $t^2$. It is also shown that the root of $\varepsilon(t)$ is always real. A theorem to validate the real roots is established.

**Keywords:** Meromorphic functions, Riemann zeta functions, zeros Riemann hypothesis.

## 1. INTRODUCTION

Let us choose

$$\varepsilon(t) = 4\int_1^\infty \frac{d(x^{3/2}\,\emptyset^1(x))}{dx}\, x^{-1/4}\cos\left(\frac{t}{2}\,log^x\right)dx \tag{1}$$

Such that

$$\emptyset(x) = \sum_{n=1}^\infty e^{-nn\pi x} \tag{2}$$

$$\emptyset'(x) = -\sum_{n=1}^\infty nn\pi\, e^{-nn\pi x} \tag{3}$$

and

$$x^{3/2}\,\emptyset'(x) = -\pi x^{3/2}\sum_{n=1}^\infty nn\, e^{-nn\pi x} \tag{4}$$

Thus

$$\frac{d(x^{3/2}\,\emptyset'(x))}{dx}\ \textit{is obtained to be}\ \frac{d}{dx}\left[-nn\pi\sum_{n=1}^\infty x^{3/2}\ e^{-nn\pi x}\right]\ \textit{and this leads to}$$

$$\pi\sum_{n=1}^\infty\left[x^{3/2}\,n^4\pi\, e^{-nn\pi x} - \frac{3n^4}{2}\,x^{1/2}\,e^{-nn\pi x}\right] \tag{5}$$

$$= \sum_{n=1}^\infty\left[x^{3/2}\,n^4\pi^2 - \frac{3n^2}{2}\,\pi\, x^{1/2}\right]\ e^{-nn\pi x} \tag{6}$$

*Address corresponding to these authors at the Department of Mathematical Sciences, Ekiti State University, Ado-Ekiti, Nigeria; Tel: 07066466859, 07037344253; E-mail: ope_taiwo3216@yahoo.com

Department of Math/Computer, College of Sciences, Federal University of Petroleum Resources, Effurun, Delta State; E-mail: adeyeyefola@yahoo.com

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If one expresses $\cos\left(\frac{1}{2}\ t\log x\right)$ in its exponential form

$$x^{-1/4}\ \cos\left(\frac{1}{2}\ t\log x\right)\ \equiv \frac{1}{2}\ x^{\frac{-1}{4}}\ e^{it/2}\ logx + \frac{1}{2}\ x^{\frac{-1}{4}}\ e^{it/2}\ logx \tag{7}$$

One can now write (1) as

$$\varepsilon(t) = \sum_{n=1}^{\infty}\left[\frac{3n^2}{4}\int_1^{\infty} x^{3/2}\ e^{\left(\frac{i\,t}{2}\ logx - nn\pi x\right)}\ dx + \frac{3n^2}{4}\int_1^{\infty} x^{1/2}\ e^{-\left(\frac{it}{2}\log x\ +\ nn\pi x\right)}\ dx \right. \\ \left. - \frac{n^2\pi^2}{2}\int_1^{\infty} x^{5/4}\ e^{\left(\frac{it}{2}\log x\ + nn\pi x\right)}\ dx - \frac{n^2\pi^2}{2}\int_1^{\infty} x^{5/4}\ e^{-\left(\frac{it}{2}\log x\ + nn\pi x\right)}\ dx\right] \tag{8}$$

One further simplification of (8), one obtains $\varepsilon(t)$ as;

$$= \sum_{n=1}^{\infty}\left[\frac{\left(\frac{3n^4\pi^2}{2x} - 3n^2\pi it\right)\ e^{-nn\pi}}{\left(3n^4\pi^2 - n^4\pi x - \frac{t^2}{2}\right) - i\left(\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)} + \frac{\left(\frac{3n^4\pi^2}{2x} + 3n^2\pi it\right)\ e^{-nn\pi}}{\left(4n^4\pi^2 - n^4\pi x - \frac{t^2}{2}\right) + i\left(\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)} \right. \\ \left. - \frac{\left(-2n^6\pi^3 + i\ \frac{n^4\pi^2}{x}\right)\ e^{-nn\pi}}{\left(4n^4\pi^2 - 5n^4\pi x - \frac{t^2}{2}\right) - i\left(5\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)} - \frac{\left(\frac{n^4\pi^2}{x} + 2n^6\pi^3\right)\ e^{-nn\pi}}{\left(4n^4\pi^2 - 5n^4\pi x - \frac{t^2}{2}\right) + i\left(5\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)}\right] \tag{9}$$

## 2. BY RATIONALIZING (9), ALL THE IMAGINARY PART VANISHES AND ONE IS LEFT WITH ONLY REAL PARTS SUCH THAT;

$$\varepsilon(t) = \frac{\left(6n^4\pi^2\ e^{-nn\pi}\right)\left(4n^4\pi^2 - n^4\pi x - \frac{t^2}{2}\right) + \left(\frac{3n^2\pi t}{x}\ e^{-nn\pi}\right)\left(\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)}{\left(4n^4\pi^2 - n^2\pi x - \frac{t^2}{2}\right)^2 + \left(\frac{xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)^2} \\ - \frac{\left(4n^6\pi^2\ e^{-nn\pi}\right)\left(4n^4\pi^2 - 5n^2\pi x - \frac{t^2}{2}\right) + \left(\frac{2n^4\pi^2}{x}\ e^{-nn\pi}\right)\left(\frac{5xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)}{\left(4n^4\pi^2 - 5n^2\pi x - \frac{t^2}{2}\right)^2 + \left(\frac{5xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)^2} \tag{10}$$

One obtains (10) as:

$$= \sum_{n=1}^{\infty}\frac{\left[\left(4n^4\pi^2 - 5n^2\pi x - \frac{t^2}{x}\right)^2 + \left(\frac{5xt}{2} - 2n^2\pi t - \frac{2n^2\pi t}{x}\right)^2\right]A - \left[\left(4n^4\pi^2 - n^2\pi x - \frac{t^2}{2}\right)^2 + \left(\frac{xt}{2} - 2n^2\pi t + \frac{2n^2\pi t}{x}\right)^2\right]B}{e^{-nn\pi}\left[\left(4n^4\pi^2 - 5n^2\pi x - \frac{t^2}{x}\right)^2 + \left(\frac{5xt}{2}\ 2n^4\pi x - \frac{2n^2\pi t}{x}\right)^2\right]\left[\left(4n^4\pi^2 - n^2\pi x - \frac{t^2}{x}\right)^2 + \left(\frac{xt}{2} - 2n^2\pi t + \frac{2n^2\pi t}{2}\right)^2\right]} \tag{11}$$

Equation (11) gives the zero of (10) by equating the numerator to zero, which can be written as;

$$\sum_{n=1}^{\infty}\left[\left(\frac{6n^4\pi^2}{x}\right)t^2+\left(\frac{6n^4\pi^2}{x}+\frac{6n^4\pi^2}{x^2}-\frac{3n^4\pi}{2}\right)t+(24n^8\pi^4-6n^6\pi^3x)\right]\left[\left(\frac{1}{x^2}\right)t^2 + \left(\frac{-8n^6\pi^3}{x}+\frac{25x^2}{4}-10n^2\pi+12n^4\pi^2+\frac{4n^4\pi^2}{x^2}\right)t^2+(16n^8\pi^4-40n^6\pi^3+25n^4\pi^2x^2)\right] + \left[\left(\frac{-4n^6\pi^3}{x^2}+5n^4\pi^2\right)t^2+16n^{10}\pi^5-20n^8\pi^4x\right]\left[\left(\frac{1}{x^2}\right)t^4 + \left(\frac{x^2}{4}-2n^2\pi x+4n^4\pi^2+\frac{4n^4\pi^2}{x^2}\right)t^2+(16n^8\pi^4-8n^6\pi^3+n^4\pi^2x^2)\right] = 0 \qquad (12) \tag{12}$$

## 3. ON FURTHER SIMPLIFICATION, ONE OBTAINS;

$$\sum_{n=1}^{\infty}[Mt^6+Nn^5+Pt^4+Qt^3+Rt^2+St+W]=0 \tag{13}$$

Such that

$$A=(6n^4\pi^2e^{-nn\pi})\left(4n^4\pi^2-n^4\pi x-\frac{t^2}{2}\right)+\left(\frac{3n^2\pi t}{x}e^{-nn\pi}\right)\left(\frac{xt}{2}-2n^2\pi t-\frac{2n^2\pi t}{x}\right) \tag{14}$$

$$B=(4n^6\pi^2e^{-nn\pi})\left(4n^4\pi^2-5n^2\pi x-\frac{t^2}{2}\right)+\left(\frac{2n^4\pi^2}{x}e^{-nn\pi}\right)\left(\frac{5xt}{2}-2n^2\pi t-\frac{2n^2\pi t}{x}\right) \tag{15}$$

$$M=\left[\left(\frac{-4n^6\pi^3}{x^2}+5n^4\pi^2\right)+\left(\frac{6n^4\pi^2}{x}\right)\right]\left(\frac{1}{x^2}\right)+\frac{5\pi^4x}{4} \tag{16}$$

$$N=\left(\frac{6n^4\pi^2}{x}+\frac{6n^4\pi^2}{x^2}-\frac{3n^2\pi}{2}\right)\left(\frac{1}{x^2}\right)-\frac{5\pi^4x}{4} \tag{17}$$

$$P=\left[\frac{44n^8\pi^4}{x^2}-\frac{48n^{10}\pi^5}{x^2}+\frac{75n^4\pi^2}{2}-\frac{66n^6\pi^3}{x}+\frac{48n^{10}\pi^4}{x}+\frac{24n^8\pi^4}{x^3}+\frac{12n^8\pi^4}{x}-n^6\pi^3-\frac{16n^{10}\pi^5}{x^5}-10n^6\pi^3x+20n^6\pi^3\right] \tag{18}$$

$$Q=\left[\frac{75xn^4\pi^2}{2}-\frac{48n^{10}\pi^5}{x^2}-\frac{60n^6\pi^3}{x}+\frac{84n^8\pi^4}{x}+\frac{24n^8\pi^4}{x^3}-\frac{48n^{10}\pi^5}{x^3}+\frac{75n^4\pi^2}{2}-n^6\pi^3-\frac{66n^6\pi^3}{x^2}+\frac{24n^8\pi^4}{x^2}-10n^6\pi^3x+20n^6\pi^3\right] \tag{19}$$

$$R=\left(12n^4\pi^2-\frac{8n^6\pi^3}{x}+\frac{25x^2}{4}-10n^2\pi+\frac{4n^4\pi^2}{x^2}\right)(24n^8\pi^4-6n^6\pi^3x)+\left(\frac{6n^4\pi^2}{x}\right)(16n^8\pi^4-40n^6\pi^3x+25n^4\pi^2x^2) +(16n^8\pi^4-8n^6\pi^3x+n^4\pi^2x^2)\left(5n^4\pi^2-\frac{4n^6\pi^3}{x^2}\right) +(16n^{10}\pi^5-20n^8\pi^4x)\left(\frac{x^2}{4}-2n^2\pi x+4n^4\pi^2+\frac{4n^4\pi^2}{x^2}\right) \tag{20}$$

$$S=(16n^8\pi^4-40n^6\pi^3x+25n^4\pi^2x^2)\left[\frac{6n^4\pi^2}{x}+\frac{6n^4\pi^2}{x^2}-\frac{3n^2\pi}{2}\right] \tag{21}$$

$$W = (16n^8\pi^4 - 8n^6\pi^3x + n^4\pi^2x^2) + (16n^{10}\pi^5 - 20n^8\pi^4x)(16n^8\pi^4 - 8n^6\pi^3x + n^4\pi^2x^2) \quad (22)$$

## 4. CONCLUSION

The solution to these polynomials are known as Algebraic function, because the function is a summation of polynomials. Hence, the solution to a polynomial is called an Algebraic number. Riemann zeta function is a function of algebraic functions; that is, it has to do with the summation of polynomials

The sum and product of all the roots of

$$a_0z^n + a_1z^{n-1} + \cdots a_n = 0$$

$a_0 \neq 0$ are $\frac{-a_1}{a_0}$ and $(-1)^n \frac{a_n}{a_0}$ respectively.

Considering our polynomials,

$$\mathrm{M}t^6 + Nt^5 + Pt^4 + Qt^3 + Rt^2 + ST + W = 0$$

$$t^6 + \frac{N}{M}t^5 + \frac{P}{M}t^4 + \frac{Q}{M}t^3 + \frac{R}{M}t^2 + \frac{S}{M}t + \frac{W}{M} = 0$$

By the theorem above, the sum of the 6 roots of the polynomials is $\frac{-N}{M}$ and the product of the roots of the polynomials is $(-1)^n \frac{W}{M}$. We are interested in using the sum of the roots to test if the roots are real or not. The result of the program shows that running *n* from 1 to 90,000 0000 and *x* running from 1 to ∞ then the sum will always be real.

Hence, this has validated the Riemann hypothesis that says all roots are real

**Theorem 1:**

Let $t_1 = a_1, t_2 = a_2, t_3 = a_3 + ib_3, t_4 = a_4, t_5 = a_5, \quad t_6 = a_6$

Then the sum of all the roots is given to be

$$t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = a_1 + a_2 + (a_3 + ib_3) + a_4 + a_5 + a_6$$

$$(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + ib$$

1. Thus if any of the roots is a complex root then the sum of the root will be a complex root
2. If all the roots are real such that

$$t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

Then the sum of all the roots will be real.

## REFERENCES

[1] On the number of Prime Number less than a given Quantity; Bernhard Riemann Translated by David R. Wilkins. Preliminary version: Dec. 1998 {Nonatsberichte der Berliner, Nov.1859}

[2] An introduction to the theory of the Riemann zeta function, by S.J.Patterson.

[3] On lower bounds for discriminants of algebraic number fields, M.Sc. thesis by S.A.Olorunsola (1980).

[4] Complex variables and Application (Third edition) By Ruel V. Churchill, James W. Brown, and Roger F. Verhey.

[5] Complex variables for scientists and engineers By John D. Paliouras.

[6] Mathematical methods for physics and engineering; A comprehensive guide by K. F. Riley, M. P. Hobson and S. J. Bence.

[7] Complex analysis (third edition) by Serge Lang; Department of mathematics Yale university New Haven, CT06520 USA. SPRINGER

[8] Problem of the millennium; hypothesis.en.wikipedia.org/wiki/ Riemann-hypo.

[9] Advanced Engineering mathematics By Erwin Kreyszig ($8^{th}$ Edition)

[10] Supercomputers and the Riemann zeta function: A.M. Odlyzko; ATandT Bell Laboratories Murray Hcll, New jersey 07974

[11] The Mathematical Unknown by John Derbyshire Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics; Joseph Henry Press, 412 pages, $24.95; Reviewer James Franklin

[12] The Riemann hypothesis by Enrico Bombieri.

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