Some New Exact Solutions for Prescribed Vorticity Distribution of Couple Stress Fluids in the Presence of Unknown Body Force

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Abstract: In the present paper, we indicate some new exact solutions of equations of motion for plane steady incompressible couple stress fluids flows in the presence of unknown body force for which vorticity distribution is defined by Eq. (12). All solutions involve arbitrary real constants and arbitrary real constants depending on the parameter θ indicating that a large number of streamfunctions and expressions for body force can be constructed. Some streamfunctions ψ are constructed for some values of parameters. The streamlines patterns of these streamfunctions are presented and discussed.

Keywords: Exact solutions, Couple stress fluid, Flows for prescribed vorticity distribution.

1. INTRODUCTION

The equations of motion of couple stress fluids are highly non-linear partial differential equations. No general method or general technique is reported in literature to solve them. However, various methods and techniques are used ***/*** reported by the researchers to solve them. The reader interested in these can refer to [1-16] and the papers cited therein. Some exact solutions to equations of motion of couple stress fluids are determined in cases where the equations can be linearized and in cases where governing equations can be reduced to ordinary differential equations for which the solution is possible. Some exact solutions are also reported by several researchers including the present author (see [14-16]) for couple stress fluids flows in which the vorticity distribution is prescribed such that the governing equations written interms of the streamfunction become linear. In obtaining these exact solutions body force and body moments are assumed to be absent. The literature available to us indicates that no exact solutions to equations governing the motion of couple stress fluids in the presence of unknown body force for prescribed vorticity distribution are presented at yet. Therefore it is of interest as fluid dynamist to investigate the solutions to the equations of motion of couple stress fluids in the presence of unknown body force for prescribed vorticity distribution.

The objective of this paper is to indicate some new exact solutions of equations describing the steady plane motion of incompressible couple stress fluids in the presence of unknown body force for which vorticity distribution is characterizes by Eq. (12). The objective is achieved by employing two functions and transformation of variables. The functions transform equations into simpler forms and the transformation of variables reduces the resulting equations into ordinary differential equations whose solutions are easily determined. The functions Ψ and H, and transformation of variables ξ are defined by Eqs. (13), (18) and (20), respectively.

The rest of the paper is organized as follows: Section (2) presents the flow equations in terms of the functions Ψ and H. Section (3) presents exact solutions. Section (4) discusses the result of section (3). Section (5) presents conclusions.

2. EQUATIONS OF MOTION

The basic equations describing the unsteady motion of incompressible couple stress fluids in the presence of body force are [2]

$$\nabla . \boldsymbol{u} = 0, \tag{1}$$

$$\rho(\boldsymbol{u}_t + (\boldsymbol{u} . \nabla)\boldsymbol{u}) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \eta \nabla^4 \boldsymbol{u} + \rho \mathbf{f}, \tag{2}$$

where $\boldsymbol{u}$ is the velocity vector, ρ is the constant density, p is the pressure, μ is the coefficient of viscosity, $\mathbf{f}$ is the body force and η is the material constant responsible for the couple stress parameter.

For steady plane flows in cartesian coordinates, Eqs. (1) and (2) can be written as

$$u_x + v_y = 0, \tag{3}$$

$$P_x - v\omega = -\nu_1 \omega_y + \nu_2 (\nabla^2 \omega)_y + F_1, \tag{4}$$

$$P_y + u\omega = \nu_1 \omega_x - \nu_2 (\nabla^2 \omega)_x + F_2, \tag{5}$$

where the generalized pressure P and the vorticity function ω are defined as

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$$P = \frac{1}{2}(u^2 + v^2) + \frac{p}{p}, \tag{6}$$

$$\omega = v_x - u_y. \tag{7}$$

The constants, $\nu_1 = \frac{\mu}{\rho}$ and $\nu_2 = \frac{\eta}{\rho}$. are the kinematic viscosity and couple stress viscosity. F_1 and F_2. are x and y components of the body force. Equation (3) implies the existence of the streamfunction ψ such that

$$u = \psi_y, \quad v = -\psi_x. \tag{8}$$

Substitution of Eq. (8) into Eqs. (4), (5) and (7), yields

$$P_x = -\psi_x\omega - \nu_1\omega_y + \nu_2(\nabla^2\omega)_y + F_1, \tag{9}$$

$$P_y = -\psi_y\omega - \nu_1\omega_x - \nu_2(\nabla^2\omega)_x + F_2, \tag{10}$$

$$\omega = -\nabla^2\psi. \tag{11}$$

We consider the motion of incompressible stress fluids in the presence of unknown body force for which the vorticity distribution is

$$\nabla^2\psi = K(\psi - Ax - By + Rxy + Cx^2 + Dy^2), \tag{12}$$

where K, A, B are real constants and R, C, D real constants depending on the parameter θ, $-\pi \le \theta \le \pi$.

Let

$$\Psi = \psi - Ax - By + Rxy + Cx^2 + Dy^2. \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain

$$\nabla^2\Psi = K\Psi + 2(C + D). \tag{14}$$

Equation (11), employing Eqs. (12) and (13), become

$$\omega = -K\Psi. \tag{15}$$

Equations (9) and (10), utilizing Eqs. (13) and (15), become

$$H_x = -K(-A + Ry + 2Cx)\Psi + (\nu_1 K - \nu_2 K^2)\Psi_y + F_1, \tag{16}$$

$$H_y = -K(-B + Rx + 2Dy)\Psi - (\nu_1 K - \nu_2 K^2)\Psi_x + F_2, \tag{17}$$

where the function $H(x, y)$ is given by

$$H(x,y) = P - \frac{1}{2}K\Psi^2. \tag{18}$$

Once a solution of Eqs. (14), (16) and (17) is determined the streamfunction ψ is determined from Eq. (13), velocity components u, v from Eq. (8), and the pressure p from Eq. (6) employing Eqs. (13) and (18).

3. SOLUTIONS

In this section we determine the exact solutions of the governing equations. We seek solutions of Eq. (14) of the form

$$\Psi(x, y) = G(\xi), \tag{19}$$

where

$$\xi = x\cos\theta + y\sin\theta, \quad -\pi \le \theta \le \pi \tag{20}$$

Inserting Eq. (19) in Eq. (14), we get

$$G_{\xi\xi} = KG + 2(C + D). \tag{21}$$

The solution of Eq. (21) is

$$G = \begin{cases} a_1(\theta)\cos(m\xi + a_2(\theta)) + \dfrac{2(C+D)}{m^2}, & \text{for } K = -m^2, m > 0, \quad (22) \\ a_3(\theta)e^{n\xi} + a_4(\theta)e^{-n\xi} - \dfrac{2(C+D)}{n^2}, & \text{for } K = -n^2, n > 0. \quad (23) \end{cases}$$

In order to determine the force components, F_1 and F_2, we apply integrability condition, $H_{xy} = H_{yx}$ on Eqs. (16) and (17) .The integrability condition yields

$$\begin{aligned}&(F_1 - 2CKx\Psi + KA\Psi)y - (F_2 - 2DKy\Psi + KB\Psi)_x \\ &\quad + (\nu_1 K - \nu_2 K^2)(K\Psi + 2(C + D)) - KR(y\Psi y - x\Psi x) = 0\end{aligned} \tag{24}$$

Since

$$y\Psi_y = (y\Psi)_y - \Psi, \tag{25}$$

$$x\Psi_x = (x\Psi)_x - \Psi, \tag{26}$$

the Eq. (24) can be rewritten as

$$\begin{aligned}&(F_1 - 2CKx\Psi - KRy\Psi + KA\Psi)_y \\ &- (F_2 - 2DKy\Psi - KRx\Psi + KB\Psi)_x \\ &\quad + (\nu_1 K - \nu_2 K^2)(K\Psi + 2(C + D)) = 0.\end{aligned} \tag{27}$$

On substituting

$$G_1 = F_1 - 2CKx\Psi - KRy\Psi + KA\Psi, \tag{28}$$

$$G_2 = F_2 - 2DKy\Psi - KRx\Psi + KB\Psi, \tag{29}$$

in Eq. (27), we obtain

$$G_{1y} = G_{2x} + (\nu_1 K - \nu_2 K^2)(K\Psi + 2(C + D)) = 0. \tag{30}$$

For solution of Eq. (30), we assume

$$G_1 = MZ(\xi), \quad G_2 = NZ(\xi) \tag{31}$$

where $M(\theta)$ and $N(\theta)$ are parametric constants.

Inserting Eq. (31) in Eq. (30), we get

$$\begin{aligned}&(M\sin\theta - N\cos\theta)Z'(\xi) + (\nu_1 K - \nu_2 K^2)\\ &\quad (KG(\xi) + 2(C+D)) = 0,\end{aligned} \tag{32}$$

whose solution is

$$Z(\xi) = \lambda(\nu_1 K - \nu_2 K^2)[K\int G\, d\xi + 2(C+D)\xi] + a_5(\theta), \tag{33}$$

where

$$\lambda = \frac{1}{m\sin\theta - N\cos\theta}. \tag{34}$$

The general solution of Eqs. (16) and (17), utilizing Eqs. (28), (29), (31) and (33), is

$$\begin{aligned}H &= -(\nu_1 K - \nu_2 K^2)\cos 2\theta \sec\theta \csc\theta G(\xi)\\ &\quad + (M\sec\theta + N\operatorname{cosec}\theta)\int Z(\xi) d)\xi + T(\theta),\end{aligned} \tag{35}$$

where $T(\theta)$ is a parametric constant.

The pressure p can easily be determined from Eq. (6) utilizing Eqs. (18) and (35) for $G(\xi)$ given by Eqs. (22) and (23). The solutions of Eqs. (3-5) can now be written down which are given below.

For $K = -m^2$, the streamfunction ψ, the velocity components u, v, the force components F_1, F_2, and pressure p are

$$\begin{aligned}
&\psi = a_1(\theta)\cos(mx\cos\theta + my\sin\theta + a_2(\theta))\\
&+\frac{2(C+D)}{m^2} + Ax + By - Rxy - Cx^2 - Dy^2\\
&u = -m\sin\theta\, a_1(\theta)\sin(mx\cos\theta + my\sin\theta\\
&+a_2(\theta)) + B - Rx - 2Dy\\
&v = m\cos\theta a_1(\theta)\sin(mx\cos\theta + my\sin\theta\\
&+a_2(\theta)) - A + Ry + 2Cx\\
&F_1 = MZ(\xi) + K(2Cx + Ry + A)\\
&\left[a_1(\theta)\cos(mx\cos\theta + my\sin\theta + a_2(\theta)) + \frac{2(C+D}{m^2}\right]\\
&F_2 = NZ(\xi) + K(2Dy + Rx + B)\\
&\left[a_1(\theta)\cos(mx\cos\theta + my\sin\theta + a_2(\theta)) + \frac{2(C+D}{m^2}\right]\\
&p = -\frac{\rho}{2}(u^2 + v^2) + \rho H + \frac{\rho K}{2}
\end{aligned}$$

where

$$\begin{aligned}
&H = -(\nu_1 K - \nu_2 K^2)\cos 2\theta \sec\theta\csc\theta G(\xi) +\\
&(M\sec\theta + N\operatorname{cosec}\theta)\int Z(\xi)\, d)\xi + T(\theta)\\
&Z(\xi) = -\lambda(\nu_1 K - \nu_2 K^2)[K\int G\, d\xi + 2(C+D)\xi] + a_5(\theta)\\
&G(\xi) = a_1(\theta)\cos(m\xi + a_2(\theta)) + \frac{2(C+D)}{m^2}\\
&\xi = x\cos\theta + y\sin\theta,\ -\pi \le \theta \le \pi.
\end{aligned}$$

For $K = n^2$, the streamfunction ψ, the velocity components u, v, the force components F_1, F_2, and pressure p are

$$\begin{aligned}
&\psi = a_3(\theta)e^{n(x\cos\theta + y\sin\theta} + a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)} - \frac{2(C+D)}{n^2}\\
&+Ax + By - Rxy - Cx^2 - Dy^2\\
&u = n\sin\theta a_3(\theta)e^{n(x\cos\theta + y\sin\theta)} - n\sin\theta\\
&a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)} + B - Rx - 2Dy\\
&v = -n\cos\theta a_3(\theta)e^{n(x\cos\theta + y\sin\theta)} + n\cos\theta\\
&a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)} - A + Ry + 2Cx\\
&F_1 = MZ(\xi) + K(2Cx + Ry + A)\\
&\left[a_3(\theta)e^{n(x\cos\theta + y\sin\theta)} + a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)} - \frac{2(C+D)}{n^2}\right]\\
&F_2 = NZ(\xi) + K(2Dy + Rx + B)\\
&\left[a_3(\theta)e^{n(x\cos\theta + y\sin\theta)} + a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)} - \frac{2(C+D)}{n^2}\right]
\end{aligned}$$

$$\begin{aligned}
&p = -\frac{\rho}{2}(u^2 + v^2) + \rho H\\
&\quad + \frac{\rho K}{2}\left[\begin{aligned}&a_3(\theta)e^{n(x\cos\theta + y\sin\theta)} + a_4(\theta)e^{-n(x\cos\theta + y\sin\theta)}\\ &-\frac{2(C+D)}{n^2}\end{aligned}\right]^2
\end{aligned}$$

where

$$\begin{aligned}
&H = -(\nu_1 K - \nu_2 K^2)\cos 2\theta \sec\theta\csc\theta\, G(\xi)\\
&+(M\sec\theta + N\operatorname{cosec}\theta)\int Z(\xi)\, d\xi + T(\theta)\\
&Z(\xi) = -\lambda(\nu_1 K - \nu_2 K^2)[K\int G\, d\xi + 2(C+D)\xi] + a_5(\theta)\\
&G(\xi) = a_3(\theta)e^{n\xi} + a_4(\theta)e^{-n\xi} - \frac{2(C+D)}{n^2}\\
&\xi = x\cos\theta + y\sin\theta,\ -\pi \le \theta \le \pi.
\end{aligned}$$

4. RESULTS AND DISCUSSION

Some new exact solutions are indicated to equations governing the steady plane incompressible couple stress fluids flows in the presence of unknown body force when vorticity distribution is prescribed by Eq. (12). The streamfunctions ψ contain arbitrary real constants K, A, B, m, n and real arbitrary constants R, C, D, a_1, a_2, a_3, a_4 depending on the parameter θ. The arbitrariness of these constants indicates that a large number of expressions for components F_1 and F_2 of body force and pressure p can be constructed through their corresponding equations. We constructed some streamfunctions ψ by assuming some expressions of parameters R, C, D, a_1, a_2, a_3 and a_4 these are

$$\psi = Ax + By - V_1\cos\theta xy - V_2\cos\theta x^2 - V_3\sin\theta y^2 + V_4\sin\theta\cos(mx\cos\theta + my\sin\theta + V_5\tan\theta) + \frac{2(V_1\cos\theta + V_2\sin\theta)}{m^2} \quad (36)$$

$$\psi = Ax + By - V_1\sin\theta xy - V_2\cos\theta x^2 - V_3\sin\theta y^2 + V_4\cos\theta\, e^{n(x\cos\theta + y\sin\theta)} + V_5\sin\theta e^{-n(x\cos\theta+y\sin\theta)} + -\frac{2(C+D)}{n^2} \quad (37)$$

where V_1, V_2, V_3, V_4 and V_5 are all non-zero real constants.

Figures (**1-7**) represents streamlines patterns for streamfunction ψ in Eq. (36) for $\theta = 20^\circ$ and various values of A, B, V_1, V_2, V_3, V_4, V_5 and m. We observe that when quadratic term $Cx^2 + Dy^2$ dominates other terms the streamlines are hyperbolic in nature, Figure **1**. If other terms dominates the quadratics term streamlines are closed curves, Figures (**2-6**). Comparing Figures (**2-4**) we find that streamlines patterns change as we increase the value of V_4 for fixed values of other constants. However the closed curve nature of the streamlines is not changed. Figures (**5-7**) depict streamlines patterns for ψ in Eq. (36) for $m = 5$. The effect of higher value of m and an increase in the value of constant V_4 on streamlines patterns is obvious for fixed values of other constants, Figures (**5** and **6**). Comparing Figures (**5** and **7**), we find that decrease in the value of V_3 changes closed curves nature of streamlines to wavy nature, Figure **7**. Figures (**7-10**) show streamlines patterns for ψ in Eq. (37) for $\theta = 60^\circ$ and various values of A, B, V_1, V_2, V_3, V_4 and V_5. The effect of change in values of constants on streamlines patterns is obvious from these figures.

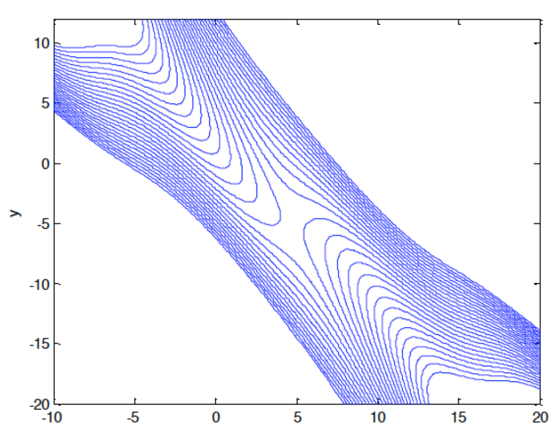

Figure 1: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 4$, $V_4 = 30$, $V_5 = 0.5$ and $m = 1$.

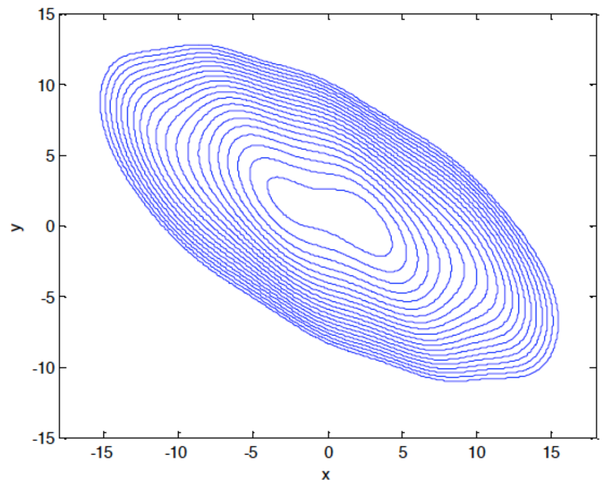

Figure 2: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 30$, $V_5 = 43$ and $m = 1$.

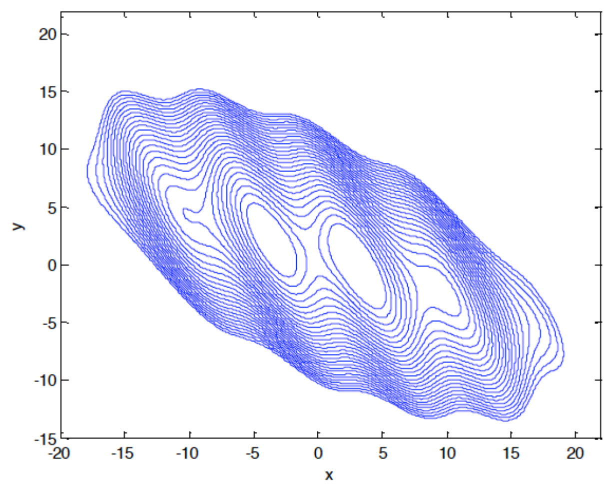

Figure 3: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 130$, $V_5 = 43$ and $m = 1$.

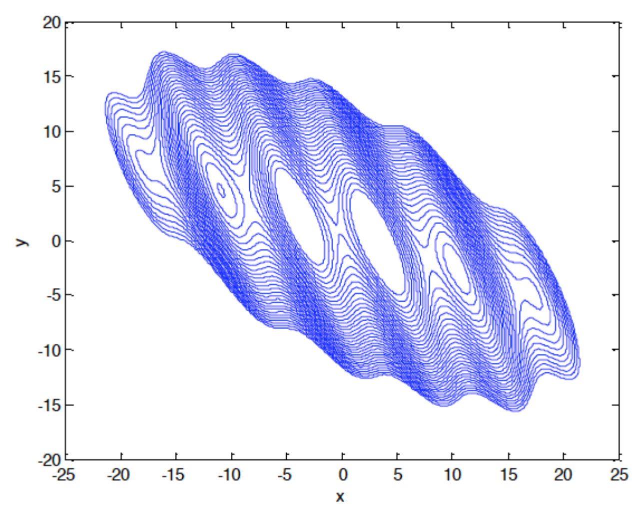

Figure 4: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 230$, $V_5 = 43$ and m = 1.

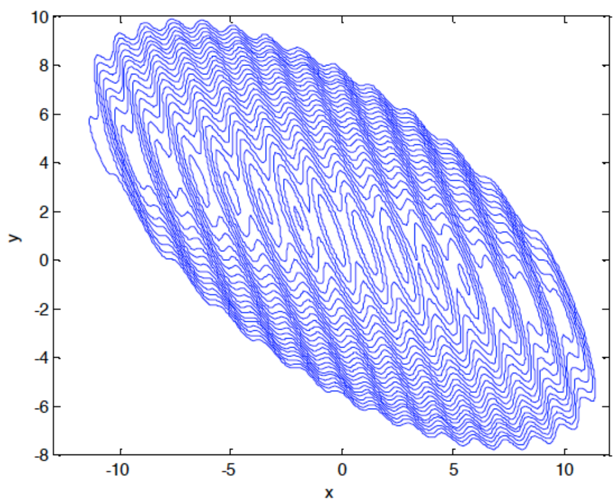

Figure 5: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 30$, $V_5 = 0.5$ and $m = 5$.

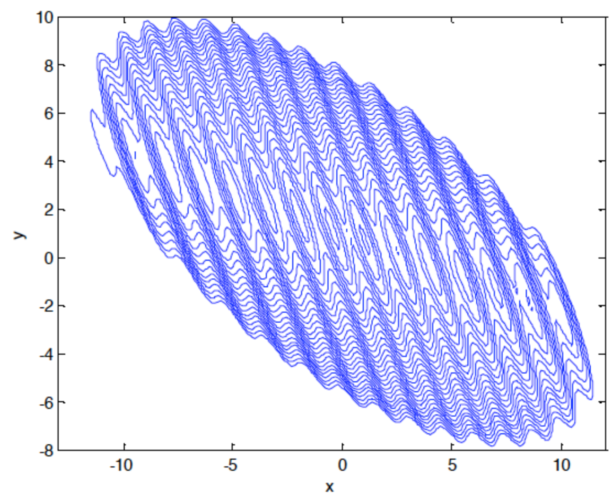

Figure 6: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 70$, $V_5 = 0.5$ and $m = 5$.

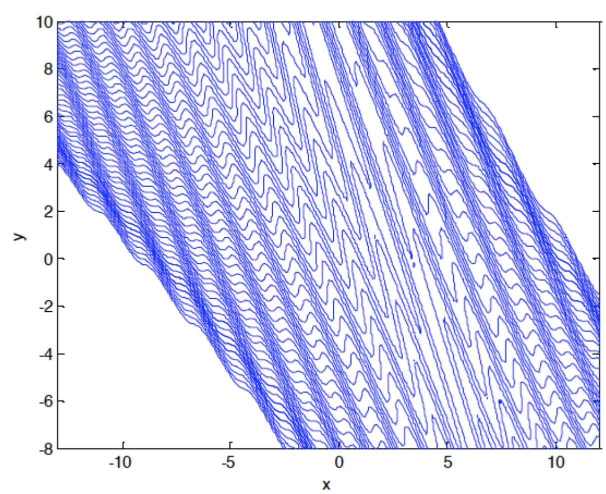

Figure 7: Streamlines pattern for ψ in equation (36) for $\theta = 20^\circ$, $A = 5$, $B = 10$, $V_1 = 5$, $V_2 = 3$, $V_3 = 14$, $V_4 = 30$, $V_5 = 0.5$ and $m = 5$.

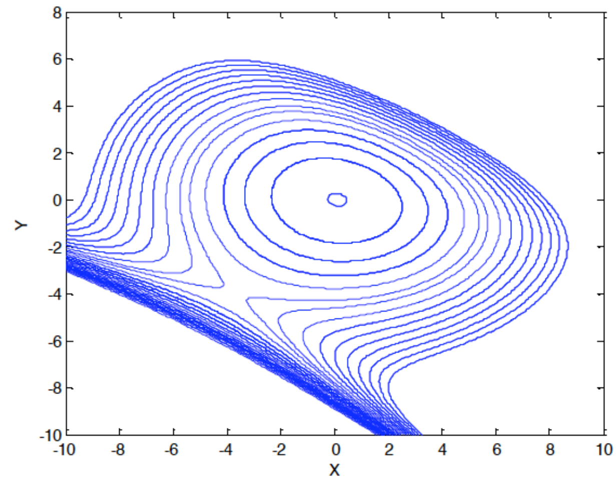

Figure 8: Streamlines pattern for ψ in equation (37) for $\theta = 60^\circ$, $A = 5$, $B = 5$, $V_1 = 2$, $V_2 = 20$, $V_3 = 20$, $V_4 = -10$, $V_5 = 1$ and $n = 1$.

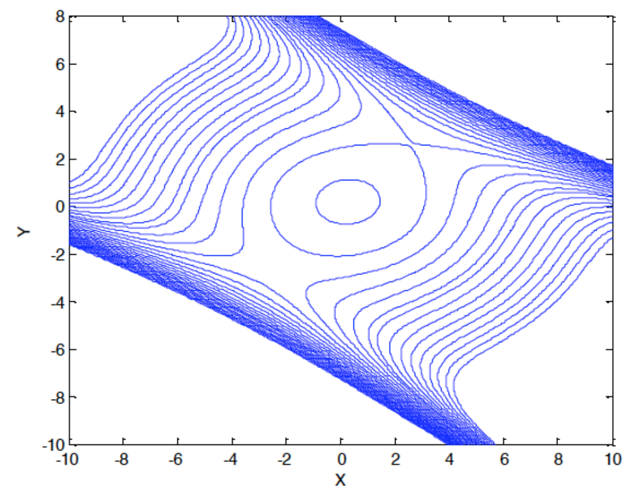

Figure 9: Streamlines pattern for ψ in equation (37) for $\theta = 60^\circ$, $A = 5$, $B = 5$, $V_1 = 2$, $V_2 = 20$, $V_3 = 20$, $V_4 = 6$, $V_5 = 4$ and $n = 1$.

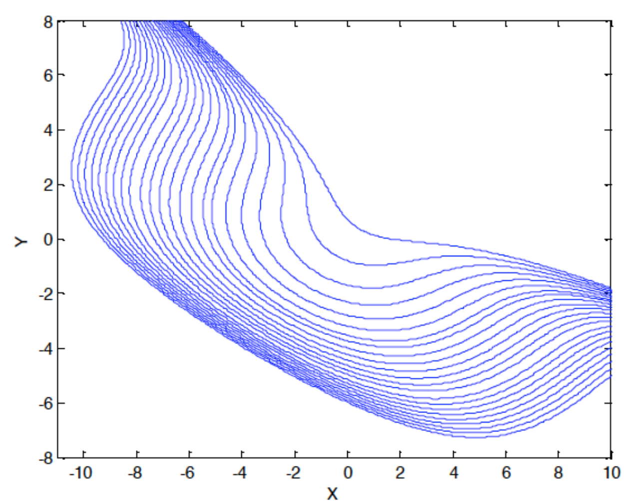

Figure 10: Streamlines pattern for ψ in equation (37) for $\theta = 60^\circ$, $A = 5$, $B = 5$, $V_1 = 2$, $V_2 = 20$, $V_3 = 20$, $V_4 = 66$, $V_5 = -4$ and $n = 1$.

5. CONCLUSIONS

In this paper some new exact solutions of equations describing the steady plane motion of incompressible couple stress fluids in the presence of unknown body force for vorticity distribution characterized by Eq. (12) are indicated. It is also indicated that due to the presence of arbitrary constants and parametric constants in streamfuntions a large number of expressions for components of body force and pressure p can be constructed. This indicates there exist a large number of solutions to flow equations. Some streamfunctions are constructed by assuming some expressions for parametric constants. The streamlines patterns for various values of real constants are presented and discussed.

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